

Polynomial Approximation Accuracy at Positive Correlation Length

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Verdict: scope-restriction (open).

1 The source statement

After Lemma 1, [MSWC24, p. 4] states that, for any $\eta > 0$, blocking

$$q = \lceil 2\xi(1 + \eta) \log N \rceil \tag{1}$$

sites gives approximation error $O(N^{-\eta})$. Here the target is the normalized periodic state of a normal tensor, the error is $\varepsilon(\psi, \phi) = 1 - |\langle \psi, \phi \rangle|$, and $\xi = -1/\log |\lambda_2|$ is the correlation length. The construction uses the gauge of eq. (5), with identity left fixed matrix and positive definite right fixed matrix ρ satisfying $\text{tr } \rho = 1$.

2 The statement proved for equal blocks

The result in `TNLean/MPS/Preparation/PolynomialAccuracy` concerns finite positive correlation length $\xi > 0$ and equal blocks. There is $C > 0$, depending on the tensor and its fixed-point data, such that for every $\eta > 0$ and all positive integers q, M with $N = Mq \geq 2$ and (1),

$$1 - \left| \langle \tilde{\phi}_N, \phi_N \rangle \right| \leq CN^{-\eta}. \tag{2}$$

The vector $\tilde{\phi}_N$ is the normalized periodic state obtained by replacing the positive part of the q -site blocked tensor by its fixed-point limit, as in eqs. (9)–(10) of the source. The constant C is independent of η, q, M .

The proof uses the stronger second-order estimate already established in `TNLean/MPS/Preparation/ApproximationError`:

$$\varepsilon(\tilde{\phi}_N, \phi_N) \leq C_\gamma M e^{-2\gamma q/\xi}, \quad 0 < \gamma < 1. \tag{3}$$

Fixing $\gamma = 1/4$ makes the constant independent of η . Since $q \geq 2\xi(1 + \eta) \log N$ and $q \geq 1$, one has $M \leq N$. The bound (2) follows from

$$M e^{-q/(2\xi)} \leq N e^{-(1+\eta) \log N} = N^{-\eta}.$$

The printed Lemma 1 gives only $O(Me^{-\gamma q/\xi})$ for $\gamma < 1/2$. Substituting (1) into that estimate gives $O(N^{1-2\gamma(1+\eta)}/q)$, whose power of N has exponent strictly greater than $-\eta$. The factor $1/q$ decays only logarithmically for fixed η . Thus that substitution alone does not establish the exact coefficient in (1); the proved second-order estimate (3) supplies one justification. The all-length argument below uses the corresponding second-order estimate for unequal blocks.

3 Every sufficiently large chain length

The equal-block proof requires $N = Mq$. For fixed η and sufficiently large prime N , the prescribed q lies strictly between 1 and N and does not divide N . This arithmetic restriction has now been removed by the actual construction in `TNLean/MPS/Preparation/AllLengthPolynomialAccuracy`.

There exist constants $C, c > 0$ and an integer $N_0 \geq 2$, chosen before η and N , such that for every $\eta > 0$ and $N \geq N_0$, the target periodic state is nonzero and the prescribed approximation $\psi_{N,\eta}$ is a unit vector satisfying

$$1 - |\langle \psi_{N,\eta}, \phi_N \rangle| \leq CN^{-\eta}, \quad (4)$$

$$T \leq c(1 + \eta) \log N. \quad (5)$$

Here T is the depth of a local circuit preparing $\psi_{N,\eta}$ from a product state. Normality supplies the eventual block injectivity and nonvanishing of the target; these are conclusions derived in the proof, not additional assumptions.

When $q \leq N$, write $N = Mq + r$, where $M = \lfloor N/q \rfloor \geq 1$ and $0 \leq r < q$. Use $M - 1$ blocks of length q and a final block of length $q' = q + r$. Thus $q \leq q' < 2q$, and their lengths sum to N . This implements the larger-final-block prescription in [MSWC24, Supplemental Material, p. 9, proof of Theorem 1, following eq. (S23)]. Apply each block's polar partial isometry to the fixed-point pairs. The resulting vector is exactly $\psi_{N,\eta}$, rather than an unspecified approximant.

The second-order unequal-block estimate is proved in `TNLean/MPS/Preparation/BlockApproximationError` for $0 < \gamma < 1$:

$$\varepsilon(\psi_{N,\eta}, \phi_N) \leq C_\gamma M e^{-2\gamma q/\xi}. \quad (6)$$

Choosing $\gamma = 1/4$ gives the same comparison $M e^{-q/(2\xi)} \leq N^{-\eta}$ as above, with one constant independent of η and N . The preparation depth is $O(q)$ because every block has length at most $2q$.

When $q > N$, the construction uses the exact normalized target. Its error is zero and exact one-block preparation has depth $O(N)$. Since $N < q$ and

$$q \leq \left(2\xi + \frac{1}{\log 2}\right) (1 + \eta) \log N \quad (N \geq 2),$$

this branch obeys the same logarithmic-depth bound. The threshold N_0 is independent of η : it can ensure both eventual nonvanishing and $2\xi \log N$ beyond every required injectivity and block-size threshold. The divisibility restriction is therefore resolved using the second-order unequal-block estimate above.

4 The remaining scope: zero correlation length

The theorem still assumes $\xi > 0$. The source’s class of normal tensors also contains tensors whose subleading transfer eigenvalues all vanish. For instance, a rank-one transfer map $\mathcal{E}(X) = \text{tr}(X)\rho$, with $\rho > 0$ of trace one, has this property. These tensors have zero correlation length and are outside the present logarithmic block prescription. The source explicitly assumes positive correlation length for its lower-bound discussion, but the upper-bound construction begins with a normal tensor without that restriction [MSWC24, pp. 2–3].

To eliminate this remaining restriction, prove that a sufficiently large, fixed positive blocking makes the transfer map equal to its fixed-point projection when the entire subleading spectrum is zero. The subleading part may be nilpotent, so one cannot assume this equality already holds at one site. Then use the resulting exact fixed-point preparation with a block size independent of N . Merely substituting $\xi = 0$ in (1) selects $q = 0$; the exact-state branch alone gives a sequential depth bound and does not establish the required logarithmic-depth conclusion.

References

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.