

The Fixed-Point State of a Tensor with Repeated Blocks

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Verdict: false-source (resolved).

1 What the source states

Let $A^i = \bigoplus_{j=1}^b \text{diag}(\mu_{j,1}, \dots, \mu_{j,m_j}) \otimes A_j^i$ with normal A_j forming a basis of normal tensors [MSWC24, eq. (S2)], block q sites, and write the polar decomposition of the blocked map as $B = VP$. The source asserts that P inherits the block structure of A ,

$$P = \bigoplus_{j=1}^b \text{diag}(\mu_{j,1}^q, \dots, \mu_{j,m_j}^q) \otimes P_j \quad (1)$$

with normal P_j [MSWC24, eq. (S5)], replaces each P_j by its fixed point, and obtains the approximating state

$$|\tilde{\phi}_N\rangle \propto V^{\otimes M} \sum_{j=1}^b \beta_j |\Omega_j\rangle, \quad \beta_j = \sum_{k=1}^{m_j} \mu_{j,k}^N, \quad |\Omega_j\rangle = \bigotimes_{k=1}^M |\omega_j\rangle_{R_k L_{k+1}}, \quad (2)$$

with $N = qM$ [MSWC24, eq. (S7)]. Part (ii) of the approximation-error lemma of the Supplemental Material then bounds $\varepsilon = 1 - |\langle \tilde{\phi}_N | \phi_N \rangle|$ by $O((N/q)e^{-\gamma q/\xi_{\text{diag}}})$ for $q = o(N)$, with ξ_{diag} the largest correlation length of the blocks.

2 The counterexample

Take $d = 2$ and two normal blocks of bond dimension one, $A_1 = (1, 0)$ with $m_1 = 2$ and $\mu_1 = (1, 1)$, and $A_2 = (0, 1)$ with $m_2 = 1$ and $\mu_2 = 1$. Then $A^0 = \text{diag}(1, 1, 0)$, $A^1 = \text{diag}(0, 0, 1)$, $D = 3$, and

$$|\phi_N\rangle \propto 2|0 \cdots 0\rangle + |1 \cdots 1\rangle, \quad \beta = (2, 1). \quad (3)$$

For every q the only nonzero blocked matrices are $B^{0 \cdots 0} = \text{diag}(1, 1, 0)$ and $B^{1 \cdots 1} = \text{diag}(0, 0, 1)$. As vectors of $\mathbb{C}^D \otimes \mathbb{C}^D$ they are $e_{00} + e_{11}$ and e_{22} , so P

has rank two with range spanned by $(e_{00} + e_{11})/\sqrt{2}$ and e_{22} , and V maps these to $|0 \cdots 0\rangle$ and $|1 \cdots 1\rangle$. On the copy index of the first block, P is the rank-one matrix $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, the square root of the restriction $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ of $B^\dagger B$, rather than $\text{diag}(\mu_{1,1}^q, \mu_{1,2}^q) = \mathbf{1}$ times a positive part, so (1) fails.

The blocks are one-dimensional, so every block has the trivial fixed point, ω_1 is e_{cc} for a copy $c \in \{0, 1\}$ and $\omega_2 = e_{22}$. Then $V e_{cc} = |0 \cdots 0\rangle/\sqrt{2}$ and $V e_{22} = |1 \cdots 1\rangle$, so

$$V^{\otimes M} \sum_j \beta_j |\Omega_j\rangle = 2 \cdot 2^{-M/2} |0 \cdots 0\rangle^{\otimes M} + |1 \cdots 1\rangle^{\otimes M}, \quad (4)$$

whose overlap with the normalized target (3) is $(2^{2-M/2} + 1)/(\sqrt{5}(2^{2-M} + 1)^{1/2})$. It equals 0.988, 0.949, 0.800 and 0.555 at $M = 1, 2, 4, 8$, tends to $1/\sqrt{5}$, and does not depend on q . Taking for instance $q \approx \sqrt{N}$, so that $q = o(N)$ and $M \rightarrow \infty$, the error tends to $1 - 1/\sqrt{5}$, while the printed bound tends to zero.

Placing the pair of the first block on both copies does not help. With $\omega_1 = (e_{00} + e_{11})/\sqrt{2}$, the two legs at one site come from different pairs, V annihilates the configurations whose two legs lie in different copies, and only the two copy-uniform configurations survive, each with amplitude 2^{-M} , so the first term is suppressed by 2^{1-M} relative to the target (3), instead of the factor $2^{-M/2}$ of (4).

No placement of the pair helps. The source does not say where ω_1 sits among the copies of the first block, so let $\omega_1 = \sum_{x,y \in \{0,1\}} W_{xy} e_{xy}$ be any unit vector on those copies, with W the 2×2 matrix of its coefficients and $\|W\|_F = 1$. The map V sends e_{00} and e_{11} to $|0 \cdots 0\rangle/\sqrt{2}$ and annihilates e_{01} and e_{10} , because $V = V\Pi$ and Π is supported on the diagonal pairs. At each site the two legs come from neighbouring pairs, so

$$V^{\otimes M} |\Omega_1\rangle = 2^{-M/2} \sum_{x \in \{0,1\}^M} \prod_{k=1}^M W_{x_k x_{k+1}} |0 \cdots 0\rangle = 2^{-M/2} \text{Tr}(W^M) |0 \cdots 0\rangle.$$

The eigenvalues λ_1, λ_2 of W satisfy $|\lambda_1|^2 + |\lambda_2|^2 \leq \|W\|_F^2 = 1$ by Schur's inequality, so $T = \text{Tr}(W^M)$ satisfies $|T| \leq 1$ for $M \geq 2$. The overlap of the approximating state with the normalized target is then $|4 \cdot 2^{-M/2} \bar{T} + 1|/(\sqrt{5}(4 \cdot 2^{-M} |T|^2 + 1)^{1/2})$, which tends to $1/\sqrt{5}$ for every choice of the pairs. The one-copy placement is $W = e_{cc}$ with $T = 1$, and the spread pair above is $W = \mathbf{1}/\sqrt{2}$ with $T = 2^{1-M/2}$. The failure is therefore not in the choice of pair: the fixed point of the positive part of the first block is not a product of nearest-neighbour pairs carrying the weight β_1 .

3 What is formalized

The fixed-point pairs of the blocks of a canonical form, each placed on one copy of its block, the state $\sum_j \alpha_j^{(N)} |\Omega_j\rangle$ and its GHZ form are formalized.¹ The counterexample is formalized for the source's construction with the pairs placed on one copy: for every $q \geq 1$ and $M \geq 1$ the overlap of (4) with the normalized target is $(4 \cdot 2^{-M/2} + 1) / (\sqrt{5} (4 \cdot 2^{-M} + 1)^{1/2})$, which tends to $1/\sqrt{5}$. The transfer maps of the two one-dimensional blocks have no eigenvalue other than one. The subleading eigenvalue of the source's definition of ξ_{diag} therefore does not exist; the formal statement reads ξ_{diag} as the correlation length at any λ_2 with $0 < |\lambda_2| < 1$ bounding the subleading eigenvalues, which here is every such λ_2 , so that $e^{-\gamma/\xi_{\text{diag}}} < 1$; along $q = M$, $N = M^2$, which satisfies $q = o(N)$, the error exceeds $C(N/q)e^{-\gamma q/\xi_{\text{diag}}} \exp(C(N/q)e^{-\gamma q/\xi_{\text{diag}}})$ for every constant C .² The positive part and the support projector of the blocked tensor are computed from the uniqueness of the positive square root of $B^\dagger B$. The computation above for an arbitrary placement of the pair of the first block is not formalized.

The hypotheses of part (ii) of the lemma are formalized for this tensor. After ordering its bond coordinates it is the direct sum (S2) of the copies of a basis of normal tensors: both blocks are normal, left-canonical with the diagonal positive-definite fixed point 1 (canonical form II), have self-overlap one, give linearly independent states for every $N \geq 1$, are not related by a gauge transformation and a phase, and give states that become orthogonal as $N \rightarrow \infty$; every weight equals one. One theorem collects these facts, with canonical form II stated for the diagonal positive-definite fixed point 1 of each block, the same fixed point from which the approximating state in its conclusion is built; it also collects the identification of the pairs (each on the first copy of its block) and of the weights β_j with those of the canonical form, and the failure of the printed bound.³

The block form (1) also fails when every multiplicity is one, as soon as the q -site states of distinct blocks are not orthogonal [con26b]. Restricting to multiplicity one therefore does not give a version of part (ii) of the lemma; restricting also to blocks whose q -site states are orthogonal does.

¹Formal declarations: `MPSTensor.SectorDecomposition.embeddedFixedPointPair` and `MPSTensor.SectorDecomposition.canonicalFixedPointState` in `TNLean/MPS/Preparation/NonNormalCanonicalForm`.

²Formal declarations: `MPSTensor.nonNormalApproxOverlap_repeatedBlockTensor`, `MPSTensor.tendsto_norm_nonNormalApproxOverlap_repeatedBlockTensor`, `MPSTensor.not_approximationError_le_repeatedBlockTensor` in `TNLean/MPS/Preparation/RepeatedBlockCounterexample`.

³Formal declarations: `MPSTensor.isBNTCanonicalForm_and_not_approximationError_le_repeatedBlock`, `MPSTensor.isBNTCanonicalForm_repeatedBlockSector`, `MPSTensor.reindex_toTensor_repeatedBlockSector` and `MPSTensor.embeddedFixedPointPair_repeatedBlockSector` in `TNLean/MPS/Preparation/RepeatedBlockCounterexample`.

4 Elimination

The printed claim stays false: with the coefficients β_j of (2) the error does not tend to zero for the tensor of the counterexample. The failure lies in the coefficients, not in the pairs. A correct statement for repeated blocks may replace $\sum_j \beta_j |\Omega_j\rangle$ by the fixed point of the actual positive part, whose restriction to the copies of block j has rank one in the copy index: on the copies of block j the positive part is close to $c_j w_j w_j^\dagger \otimes P_j$ with $c_j = (\sum_k |\mu_{j,k}|^{2q})^{1/2}$ and a unit vector w_j built from the weights $\mu_{j,k}^q$, and the corrected approximating state is $V^{\otimes M} \sum_j \beta_j L_j^{\otimes M} |\Omega_j\rangle$, with the source's weights β_j and an on-site isometry L_j that places the pairs of block j on all of its copies at once [con26a]. When the q -site states of distinct blocks overlap, the rate must also include the correlation lengths of the mixed transfer maps [con26b].

5 The source's state with corrected coefficients

The source's construction itself, with the pairs of block j on one copy k_j , converges once its coefficients are changed. Let $K_{j,k}$ place the bond pairs of copy k of block j . The blocked map is $B = \sum_j c_j B_j L_j^\dagger$ with $L_j = \sum_k (\mu_{j,k}^q / c_j) K_{j,k}$, so $BK_{j,k} = \mu_{j,k}^q B_j = (\mu_{j,k}^q / c_j) BL_j$, and since the partial isometry V vanishes on the kernel of B ,

$$VK_{j,k} = \frac{\mu_{j,k}^q}{c_j} VL_j.$$

Hence $V^{\otimes M}$ sends the pairs on copy k_j to $(\mu_{j,k_j}^q / c_j)^M$ times the pairs placed by L_j , and the state (2) with the coefficients

$$\beta'_j = \beta_j \left(\frac{c_j}{\mu_{j,k_j}^q} \right)^M$$

in place of β_j is a positive multiple of the corrected state, for weights of any phase and with no orthogonality of the blocks. It therefore has the same error. For the counterexample above the coefficients are $(2 \cdot 2^{M/2}, 1)$, and the state (4) becomes $2|0 \dots 0\rangle^{\otimes M} + |1 \dots 1\rangle^{\otimes M}$, the target. For a block of multiplicity one, $\beta'_j = |\mu_{j,k_j}|^N$.

Let the blocks be normal in the gauge $\sum_i (A_j^i)^\dagger A_j^i = \mathbf{1}$ with positive definite fixed points σ_j of trace one, from which the pairs are built, let λ_2 bound the moduli of the eigenvalues other than one of every transfer matrix of a block and of all eigenvalues of the mixed transfer matrices of distinct blocks, let $\xi = -1/\ln|\lambda_2|$, and let $0 < \gamma < 1$. There are $C, C' > 0$, independent of the weights, such that for all nonzero complex weights, every choice of the copies k_j , every $q \geq 1$ and every $M \geq 1$ with $\beta \neq 0$, the source's state with the coefficients β'_j satisfies

$$\varepsilon \leq Cy e^{Cy}, \quad \varepsilon \leq C'y, \quad y = M e^{-\gamma q / \xi}.$$

This is part (ii) of the lemma for the source's construction with the coefficients β'_j , with ξ_{diag} replaced by $\max(\xi_{\text{diag}}, \xi_{\text{off-diag}})$ [con26b], and with no factor depending on the weights [con26c]. When the q -site states of distinct blocks are orthogonal at the length q , the same bounds hold with λ_2 bounding only the eigenvalues of the transfer matrices of the blocks: orthogonality at one length makes the mixed transfer matrices nilpotent.

6 What is formalized of the elimination

The coefficients β'_j , the identity $VK_{j,k} = (\mu_{j,k}^q/c_j)VL_j$, the identification of the source's state with the coefficients β'_j with the corrected state, the equality of their overlaps with the target, and the error bounds above, for blocks whose q -site states are orthogonal and for blocks whose states overlap, are formalized.⁴ The evaluation of the corrected coefficients on the counterexample above is not formalized separately; it is a special case of the identity with the corrected state.

References

- [con26a] The TNLean contributors. A corrected approximating state for a tensor with repeated blocks. Paper-gap note mswc24_repeated_block_corrected_state, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/mswc24_repeated_block_corrected_state.pdf.
- [con26b] The TNLean contributors. The positive part of a blocked tensor with overlapping blocks. Paper-gap note mswc24_block_form_mixed_overlap, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/mswc24_block_form_mixed_overlap.pdf.
- [con26c] The TNLean contributors. Repeated overlapping blocks with a small weight sum. Paper-gap note mswc24_repeated_overlap_small_weight_sum, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/mswc24_repeated_overlap_small_weight_sum.pdf.

⁴Formal declarations: `MPSTensor.oneCopyWeight`, `MPSTensor.oneCopyWeight_of_subsingleton`, `MPSTensor.polarIso_blockTensor_repeatedBlockSum_mul_pairEmbedding`, `MPSTensor.nonNormalApproxVector_repeatedBlockSum_oneCopy`, `MPSTensor.nonNormalApproxOverlap_repeatedBlockSum_oneCopy`, `MPSTensor.exists_approximationError_le_repeatedOverlappingBlockSum_oneCopy` and `MPSTensor.exists_approximationError_r_le_mul_repeatedOverlappingBlockSum_oneCopy` in `TNLean/MPS/Preparation/OneCopyPairState`; `MPSTensor.exists_approximationError_le_repeatedBlockSum_oneCopy` and `MPSTensor.exists_approximationError_le_mul_repeatedBlockSum_oneCopy` in `TNLean/MPS/Preparation/BlockSumError`.

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.