

The Full-Input Pseudoinverse Obstruction and Its Support Correction

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Verdict: false-source (resolved).

1 Source and the full-input obstruction

For a blocked matrix product map $B = VP$, arXiv:2307.01696, page 3, footnotes 3 and 4 to equations (13)–(15), allows B to be non-injective, interprets the inverse of P as its pseudoinverse, and permits the input to be placed at any chosen tensor, with sequential decompositions from both boundaries toward that site.

If B is injective, then $V^\dagger V = \mathbb{1}_{D^2}$ and the inward sweeps produce a central isometry on the full virtual-pair input. For non-injective B , its polar support Π instead satisfies $V^\dagger V = \Pi$. Thus a central map on all D^2 input coordinates cannot remain an isometry when $\Pi \neq \mathbb{1}_{D^2}$. The assertion that the full-input isometry identity in equation (15) remains valid under footnote 3 is therefore false as printed. Restricting the domain to the support corrects the assertion; it does not prove that full-input identity.

2 A nonzero rank-deficient witness

Take $d = D = 2$ and the two nonzero matrices

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1)$$

On a ring these give the nonzero vector $|0\rangle^{\otimes N} + |1\rangle^{\otimes N}$. Block two sites, and order both the physical and virtual pairs as 00, 01, 10, 11. Since $B_{ij,ab} = (A^i A^j)_{ab}$, the blocked map and its canonical polar factors are

$$B = P = P^+ = V = \Pi = \text{diag}(1, 0, 0, 1). \quad (2)$$

Thus $d^2 = D^2 = 4$, both tensor components are nonzero, and $V^\dagger V = \Pi \neq \mathbb{1}_4$. The pseudoinverse construction already fails on these nondegenerate data. One may separately complete the partial isometry on its kernel, but that completion is not BP^+ and is not the construction asserted in equation (13).

3 A nonzero normal counterexample

Take one physical site with $d = 4$ and bond dimension $D = 2$, so the capacity condition $d^q \geq D^2$ in footnote 1 holds with equality. With $a = 1/\sqrt{2}$, let

$$B^0 = a\mathbb{1}, \quad B^1 = aE_{01}, \quad B^2 = aE_{10}, \quad B^3 = 0. \quad (3)$$

This tensor is nonzero, unital and trace preserving. Its length-two products include $E_{00}/2, E_{01}/2, E_{10}/2, E_{11}/2$, so it is normal. The one-site physical matrix, in virtual-pair order 00, 01, 10, 11, is

$$B = \begin{pmatrix} a & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V = BP^+ = \begin{pmatrix} a & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4)$$

For the unit vector $w = (1, 0, 0, -1)^\top/\sqrt{2}$, one has $Vw = 0$; indeed $V^\dagger V = \mathbb{1} - |w\rangle\langle w| \neq \mathbb{1}$. Thus neither a zero tensor nor an insufficient output dimension accounts for the obstruction. The formal witness is `MPSPreparation.mixedPolarFullInputCounterexample_spec`. Its proof uses the zero final physical row and the actual polar right-factor identity $V = BG$, without assuming a formula for a shadow polar map.

An isometric extension on the unused input direction can exist here, but it differs from BP^+ . The counterexample concerns that printed pseudoinverse identification, not the existence of an extension or the injective-block preparation theorem.

4 The support construction

Choose orthonormal coordinates for the range of Π : an isometry $J: \mathbb{C}^s \rightarrow \mathbb{C}^{D^2}$ with $JJ^\dagger = \Pi$. The bound $s \leq D^2$ follows from injectivity of J . Then

$$(VJ)^\dagger(VJ) = J^\dagger \Pi J = \mathbb{1}_s. \quad (5)$$

The existing polar right-factor theorem gives $V = BG$ for a virtual-pair matrix G . Attach GJ at the chosen central site and run the two independent inward sweeps. Their outer evaluations are isometries E_L, E_R , and their remainders give a central map C such that

$$VJ = (E_L \otimes \mathbb{1}_d \otimes E_R)C, \quad C^\dagger C = \mathbb{1}_s. \quad (6)$$

Both outer bonds have dimension one, and every side bond has dimension at most D^2 . The input may be attached at either endpoint or any interior site; the argument also allows zero-dimensional support.

The original polar partial isometry is recovered exactly:

$$(VJ)J^\dagger = V\Pi = V. \quad (7)$$

Consequently the blocked tensor is $B = (VJ)(J^\dagger P)$. The central operation is an actual isometry on the compressed input, and no physical amplitude is discarded by choosing those input coordinates.

5 Conclusion

The mixed factorization now covers every blocked tensor on its actual polar support, with no injectivity assumption. The full-input theorems remain useful specializations for injective blocks. The same mixed factorization predicate handles arbitrary finite input dimension, so no second isometry notion or support representation is introduced. The corrected support-domain theorem is proved, so the local repair is resolved. The false-source record and the witness in (2) remain: the printed full-input pseudoinverse claim is not established, and the unreduced polar map is only a partial isometry.¹

¹Formal declarations: `MPSPreparation.exists_mixed_sequential_polar_support_of_split` and `MPSPreparation.exists_mixed_sequential_polar_support` in `TNLean/MPS/Preparation/MixedSequentialSupport`.