

GHZ Form of the Fixed-Point State: Local Orthogonality as a Hypothesis

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Verdict: scope-restriction (open).

1 What the source states

Let $A^i = \bigoplus_{j=1}^b \text{diag}(\mu_{j,1}, \dots, \mu_{j,m_j}) \otimes A_j^i$ be a basis of normal tensors, let $|\omega_j\rangle$ be the fixed-point pair of the normal block A_j , embedded in the bond space $\mathbb{C}^D \otimes \mathbb{C}^D$ of A , and put

$$|\Omega'\rangle = \sum_{j=1}^b \alpha_j \bigotimes_{k=1}^M |\omega_j\rangle_{R_k L_{k+1}}. \quad (1)$$

After eq. (S7) of [MSWC24] the pairs are asserted to be locally orthogonal,

$$\langle \omega_j | \omega_{j'} \rangle = \delta_{jj'}, \quad (2)$$

with a reference to the canonical-form theory of matrix product density operators. The paragraph after eq. (19) then concludes that $W: |j\rangle \mapsto |\omega_j\rangle$ is an isometry and that $|\Omega'\rangle = W^{\otimes M} |\chi_M\rangle$ with $|\chi_M\rangle = \sum_j \alpha_j |j\rangle^{\otimes M}$. In the source, (2) is a property of these particular pairs, not a hypothesis.

2 The restriction adopted

The formal statements are made for an arbitrary family $\omega_1, \dots, \omega_b \in \mathbb{C}^D \otimes \mathbb{C}^D$ and arbitrary coefficients α .¹ The identity $|\Omega'\rangle = W^{\otimes M} |\chi_M\rangle$ holds for every such family, since $W^{\otimes M} |j\rangle^{\otimes M} = \bigotimes_k |\omega_j\rangle$. The isometry of W and the normalization $\langle \Omega' | \Omega' \rangle = 1$ take (2) as a hypothesis. For the pairs of the source they are therefore proved only once (2) is supplied.

One half of the hypothesis is available: if the components $\omega_{(a,c)}$ vanish for $a \notin S$ and the components $\omega'_{(a,c)}$ vanish for $a \notin S'$, where S and S' are disjoint

¹Formal declarations: `MPSTensor.isIsometry_pairIsometry`, `MPSTensor.nonNormalFixedPointState_eq_tensorPower_mulVec_ghzState` and `MPSTensor.nonNormalFixedPointState_norm_sq` in `TNLean/MPS/Preparation/NonNormalFixedPoint`.

sets of virtual indices, then $\langle \omega | \omega' \rangle = 0$.² The embedded pairs of distinct blocks are supported in this way.

3 A discharged instance

The hypothesis is discharged for one identification of the pairs, which the source does not make.³ The canonical form is taken as a sector decomposition, whose bond space is $\mathbb{C}^D$ with $D = \sum_j m_j D_j$ and whose blocks carry no normality or gauge condition. The pair of block j is the fixed-point pair of a positive semidefinite, unit-trace matrix σ_j on $\mathbb{C}^{D_j}$, a free parameter, placed on both legs on the bond coordinates of one chosen copy k_j of that block. The coordinates of distinct blocks are distinct, so the right legs of the pairs of distinct blocks have disjoint supports and the pairs are orthogonal by the disjoint-support lemma; the coordinates of one copy are distinct, so $\langle \omega_j | \omega_j \rangle = \text{Tr } \sigma_j = 1$. For these pairs (2) holds, and the isometry of W , the GHZ form and the normalization hold without the hypothesis.

This orthogonality follows from the placement alone. It uses neither the normality of the blocks nor the basis-of-normal-tensors property, which is the reason the source gives. The source's pairs are the fixed points of the positive parts P_j of the blocked tensor, and the block form of the positive part that would identify them with the pairs above is not formalized. When some block has multiplicity $m_j \geq 2$ that block form fails and no product of pairs carrying the weights β_j gives the approximating state [con26].

4 Elimination

Formalize the block form of the positive part of the blocked tensor for $m_j = 1$, identify its fixed-point pairs with the embedded pairs above, and derive (2) for them; the case $m_j \geq 2$ is governed by the false-source record [con26].

References

- [con26] The TNLearn contributors. The fixed-point state of a tensor with repeated blocks. Paper-gap note mswc24_multiplicity_fixed_point, TNLearn, 2026. https://sirui-lu.com/TNLearn/paper-gaps/mswc24_multiplicity_fixed_point.pdf.

²Formal declaration: `MPSTensor.inner_pair_eq_zero_of_disjoint` in the same module.

³Formal declarations: `MPSTensor.SectorDecomposition.inner_embeddedFixedPointPair`, `MPSTensor.SectorDecomposition.isIsometry_pairIsometry_embeddedFixedPointPair`, `MPSTensor.SectorDecomposition.canonicalFixedPointState_eq_tensorPower_mulVec_ghzState` and `MPSTensor.SectorDecomposition.canonicalFixedPointState_norm_sq` in `TNLearn/MPS/Preparation/NonNormalCanonicalForm`.

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.