

Preparation in Logarithmic Depth: the Periodic State Must Not Vanish

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Verdict: local-correction (resolved).

1 What the source states

Let A be a normal tensor of physical dimension d and bond dimension D . Equation (1) of [MSWC24] states that the normalized translation-invariant state $|\phi_N\rangle$ generated by A on a ring of N sites is prepared with error ε by a local circuit of depth $O(\log(N/\varepsilon))$. The Supplemental Material writes the state as $|\phi_N\rangle = c_N^{-1} \sum_{i_1, \dots, i_N} \text{Tr}(A^{i_1} \dots A^{i_N}) |i_1 \dots i_N\rangle$, “where $c_N > 0$ is the normalization constant”, and its proof of Theorem 1 begins “Let $\{|\phi_N\rangle\}$ be a sequence of TI normalized MPS on N sites generated by a normal tensor A ”. Thus $c_N = \|\phi_N(A)\|$ with $\phi_N(A)(s) = \text{Tr}(A^{s_1} \dots A^{s_N})$, and the source takes $c_N > 0$.

2 The degenerate case

The condition $c_N > 0$, that is $|\phi_N(A)\rangle \neq 0$, can fail for a normal tensor at small N . Take $d = 4$, $D = 3$, $\zeta = e^{2\pi i/3}$, and

$$A^1 = E_{12}, \quad A^2 = E_{23}, \quad A^3 = E_{31}, \quad A^4 = \text{diag}(1, \zeta, \zeta^2).$$

Every product of two of these matrices has zero trace: a product containing an off-diagonal matrix unit is off-diagonal or zero, and $\text{Tr}((A^4)^2) = 1 + \zeta^2 + \zeta^4 = 0$. Hence $|\phi_2(A)\rangle = 0$. The words of length three span $M_3(\mathbb{C})$: the words with three off-diagonal letters give the diagonal matrix units, those with one off-diagonal letter give nonzero multiples of E_{12}, E_{23}, E_{31} , and those with two give nonzero multiples of E_{13}, E_{21}, E_{32} . So A is normal.

For such N there is no normalized state. With the convention that the normalization of the zero vector is zero, every unit vector has error 1, and the claim fails for $\varepsilon < 1$.

3 The correction adopted

The formal statement assumes $|\phi_N(A)\rangle \neq 0$, the source's $c_N > 0$.¹ For every normal tensor there is N_0 with $|\phi_N(A)\rangle \neq 0$ for all $N \geq N_0$: in the gauge of eq. (5) of the source, $c_N^2 = \text{Tr} \mathcal{E}_A^N$ converges to 1; the source's proof of Lemma 1'(i) uses the same identity, " $c_N = \sqrt{\text{tr}(E_A^N)}$ ".² Equation (1) is an asymptotic statement, and for $N \geq N_0$ the assumption is not needed.

References

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.

¹Formal declaration: `MPSPreparation.exists_isPreparedInDepth_le_log_of_mpvState_ne_zero` in `TNLean/MPS/Preparation/LogDepthPreparation`.

²Formal declarations: `MPSPreparation.exists_mpvState_ne_zero_of_le` and the resulting statement without the assumption, `MPSPreparation.exists_isPreparedInDepth_le_log`.