

Preparation in Logarithmic Depth: Chain Lengths Divisible by the Block Length

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Verdict: scope-restriction (resolved).

1 What the source states

Let A be a normal tensor of physical dimension d and bond dimension D , and let $|\phi_N\rangle$ be the normalized translation-invariant state it generates on a ring of N sites. Equation (1) of [MSWC24] states that $|\phi_N\rangle$ is prepared with error ε by a circuit of local gates of depth

$$T = O(\log(N/\varepsilon)). \quad (1)$$

The error of a unit vector $|\psi\rangle$ is $\varepsilon(\psi, \phi_N) = 1 - |\langle\psi|\phi_N\rangle|$. The argument blocks q sites, prepares the approximating state $|\tilde{\phi}_N\rangle = (\bigotimes_{i=1}^{N/q} U_i)|\Omega\rangle$ of eq. (10) of the source in depth $O(q)$, bounds its error by $O((N/q)e^{-\gamma q/\xi})$ (Lemma 1), and concludes that $q = O(\log(N/\varepsilon))$ suffices. Equation (10) has N/q blocks of q sites, so the argument as written takes q to divide N . The proof of the lower bound in the Supplemental Material of the source, which uses the same blocking, says instead that the blocks are “all of the same size, q_N , except for the last one, which may be larger”.

2 The restriction formerly adopted

The formal statement first asked that the block length divide the chain length.¹ There is C , depending only on d and D , such that for every normal tensor A with $D \geq 1$ there are $a > 0$ and $b \geq 1$, depending only on A , with the following property. For $0 < \varepsilon \leq 1$, every $N \geq 1$ and every q dividing N with $q \geq a \log(N/\varepsilon) + b$, a unit vector $|\psi\rangle$ with $\varepsilon(\psi, \phi_N) \leq \varepsilon$ is prepared in depth at most Cq . If moreover $N \geq 2$ and $q \leq 2(a \log(N/\varepsilon) + b)$, for instance $q = \lceil a \log(N/\varepsilon) + b \rceil$ when it divides N , the depth is at most $c \log(N/\varepsilon)$ with

¹Formal declarations: `MPSPreparation.exists_isPreparedInDepth_approximationError_le` and `MPSPreparation.exists_isPreparedInDepth_le_log_of_dvd` in `TNLean/MPS/Preparation/DepthLogBound`.

c depending only on A . This is (1) for those pairs (N, ε) for which such a q divides N . For other chain lengths, such as prime N larger than the admissible block lengths, the statement says nothing.

3 Elimination

Allow the last block to be larger, as in the proof of the lower bound: $N = (M - 1)q + q'$ with $q \leq q' < 2q$. Three steps generalize.

1. The approximating state is the product of the isometries V_q on the first $M - 1$ blocks and $V_{q'}$ on the last block, applied to the same product of pairs $|\omega\rangle$.
2. Its overlap with $|\phi_N\rangle$ is the trace of the product of $M - 1$ mixed transfer matrices of P_q and one of $P_{q'}$ against the fixed point, and the telescoping bound of Lemma 1 applies to a product of matrices each within $O(e^{-\gamma q/\xi})$ of the same idempotent.
3. The circuit applies the block unitary of length q' to the last block; its depth is $O(q') = O(q)$.

With these, (1) holds for every $N \geq 2$ with $N \geq q$, where $q = \lceil a \log(N/\varepsilon) + b \rceil$. Chains shorter than this block length, $N < q$, have no block of length q and are not covered by the three steps; they need a separate argument. The source does not treat them: its proof of the lower bound takes N “such that we have a large number of blocks”.

4 Resolution

The restriction is removed by the unequal blocks of the proof of the lower bound, together with an exact preparation of chains shorter than the block length.² Let $q = \lceil a \log(N/\varepsilon) + b \rceil$.

Chains with $N \geq q$. Write $N = (M - 1)q + q'$ with $M = \lfloor N/q \rfloor$ and $q' = q + (N \bmod q)$, so $q \leq q' < 2q$. The three steps of the elimination plan above hold for any lengths $\ell_1, \dots, \ell_M$ of the blocks. The state $(\bigotimes_k V_{\ell_k})|\Omega\rangle$ is a unit vector when the blocked tensors are injective. Its overlap with $|\phi_N(A)\rangle$ is $\text{Tr}(\tau_{\ell_1} \cdots \tau_{\ell_M})$, with τ_ℓ the mixed transfer matrix of P_ℓ against P_∞ . The

²Formal declarations: `MPSPreparation.exists_isPreparedInDepth_le_log_of_mpvState_ne_zero` and `MPSPreparation.exists_isPreparedInDepth_le_log` in `TNLean/MPS/Preparation/LogDepthPreparation`; the three steps are `MPSTensor.blockIsometryState`, `MPSTensor.exists_blockApproximationError_le_mul`, and `MPSPreparation.exists_isPreparedInDepth_blockIsometryState`; the short chains are `MPSPreparation.exists_isPreparedInDepth_normalizedMPVState` and `MPSPreparation.exists_isPreparedInDepth_of_norm_eq_one`.

telescoping identity

$$X_1 \cdots X_M - E^M = \sum_{k=1}^M X_1 \cdots X_{k-1} (X_k - E) E^{M-k} \quad (2)$$

bounds it for any matrices with $\|X_k - E\| \leq \delta$ and $E^2 = E$, by $c((1 + c\delta)^M - 1)$ as for equal factors. With $\|\tau_\ell - \tau_\infty\| = O(e^{-\gamma\ell/\xi}) \leq O(e^{-\gamma q/\xi})$ for $\ell \geq q$ and $|\lambda_2| \leq 1$, the error is $O(Me^{-\gamma q/\xi})$, which is at most ε as for equal blocks. The circuit applies a staircase of depth $O(\ell_k)$ on each block, in parallel; the depth is $O(q') = O(q)$.

Chains with $N < q$. Then $N < a \log(N/\varepsilon) + b$, and it suffices to prepare $|\phi_N\rangle$ exactly in depth $O(N)$. When the N -site blocked tensor $\mathcal{B}_N = VP$ is injective and $N \geq 3D$, $\phi_N(A)(s) = \text{Tr } \mathcal{B}_N^s = \sum_{l,r} \langle s|V|l,r\rangle \text{Tr } P^{(l,r)}$, so $|\phi_N\rangle$ is the state of a single block with the pair $\omega(r, l) \propto \text{Tr } P^{(l,r)}$ in place of the pair of the fixed point, and the same circuit prepares it in depth $O(N)$. This is the sequential preparation of a matrix product state. Below the injectivity length plus $3D$, which depends only on A , every unit vector is a unitary applied to $|0 \cdots 0\rangle$ on a bounded number of sites, a circuit of bounded depth. In both regimes the depth is at most a constant times $a \log(N/\varepsilon) + b \leq (a + b/\log 2) \log(N/\varepsilon)$.

The normalization of $|\phi_N(A)\rangle$ presupposes $|\phi_N(A)\rangle \neq 0$; the main theorem assumes it, and it holds for every sufficiently large N ; see `docs/paper-gaps/mswc24_depth_upper_bound_nonzero_state.tex`.

References

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.