

The Depth Lower Bound in the Gauge of a Normal Tensor

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Verdict: scope-restriction (resolved).

1 What the source states

Let $|\phi_N\rangle$ be the normalized translation-invariant matrix product states on N sites generated by a normal tensor A with correlation length $\xi = -1/\ln|\lambda_2| > 0$, and let $|\psi_N\rangle$ be obtained from product states by local circuits of depth T . Theorem 1 of [MSWC24] states that if $T = o(\log N)$, then $\varepsilon(\phi_N, \psi_N) = 1 - |\langle\phi_N|\psi_N\rangle| > 1/2$ for all N larger than some N_0 .¹ The source’s definition of a normal tensor includes the normalization $\lambda_1 = 1$ of the leading eigenvalue of the transfer map $\mathcal{E}_A$, and the source remarks that after a gauge transformation $\mathcal{E}_A = |\rho\rangle\langle\mathbb{K}| + R$, that is,

$$\sum_i A^{i\dagger} A^i = \mathbb{K}, \quad \mathcal{E}_A(\rho) = \rho, \quad \rho > 0, \quad \text{Tr } \rho = 1. \quad (1)$$

2 What is formalized

The gauge-form statement assumes (1), a length $L \geq 1$ at which the products of L matrices of A span the full matrix algebra, and an eigenvalue $\lambda_2 \neq 1$ of $\mathcal{E}_A$ whose modulus bounds those of all other eigenvalues different from 1, with $-1/\ln|\lambda_2| > 0$. Under these hypotheses it proves the source’s conclusion, and the quantitative form: there are $B \geq 0$ and $C > 0$ such that $|\langle\phi_N|\psi\rangle| \geq 1/2$ for a unit vector ψ prepared in depth T and $N \geq B(T+1)$ forces $N \leq C(T+1)e^{4T/\xi}$; the exponent is discussed in `docs/paper-gaps/mswc24_depth_lower_bound_constant.tex`.²

The restriction was the gauge (1). Both the source’s proof and the chapter’s proof begin by passing to it: a normal tensor becomes $\lambda_1^{-1/2} \sigma^{1/2} A^i \sigma^{-1/2}$ with

¹Source traceability: label `th:1` of the arXiv LaTeX source, restated in the Supplemental Material, section “Proof of Theorem 1”; source at <https://arxiv.org/abs/2307.01696>.

²Formal declarations: `MPSTensor.eventually_one_half_lt_infidelity_of_isLittle0_log` and `MPSTensor.exists_depth_lower_bound` in `TNLean/MPS/Preparation/DepthLowerBound`.

σ the positive fixed point of the dual transfer map, which multiplies $|\phi_N(A)\rangle$ by a nonzero constant and leaves the normalized vector and the correlation length unchanged.

3 Resolution

The reduction to the gauge is now formalized, and the source’s Theorem 1 is proved without the gauge hypothesis.³ For a normal tensor in the source’s sense (irreducible, with $\mathcal{E}_A$ having the unique largest eigenvalue $\lambda_1 = 1$), a gauge transformation alone reaches (1); it leaves every trace $\text{Tr}(A^{i_1} \dots A^{i_N})$ and every eigenvalue of $\mathcal{E}_A$ unchanged, and some blocking of the gauged tensor is injective. For a tensor some blocking of which is injective, with no normalization of the leading eigenvalue λ_1 of $\mathcal{E}_A$, a rescaling by $\zeta \neq 0$ followed by a gauge transformation reaches (1); it multiplies $|\phi_N(A)\rangle$ by ζ^N , so $|\langle\phi_N|\psi\rangle|$ is unchanged for the normalized vectors, and it divides every eigenvalue of $\mathcal{E}_A$ by λ_1 , so the correlation length $-1/\ln|\lambda_2/\lambda_1|$ is the one of the gauged tensor. The equality $|\zeta|^2\lambda_1 = 1$ follows because $|\zeta|^2\lambda_1$ is an eigenvalue of largest modulus of a trace-preserving primitive transfer map with fixed point $\rho > 0$. Both statements are then the gauge form applied to the gauged tensor.

4 Proof route

The formal proof follows the chapter’s argument rather than the source’s. The source bounds a fidelity of blocked product states and uses its Lemma 2 at the single separation $s = 2T + 1$ with vanishing one-point functions; the correction of that lemma is recorded in `docs/paper-gaps/mswc24_decaying_correlations_windowed_connected.tex`. The chapter instead averages the centred observables over $n \approx N/\Delta$ windows spaced $\Delta = s' + L + 2T$ apart and compares the averages in $|\phi_N\rangle$ and $|\psi_N\rangle$ through the inequality $|\langle\phi|\psi\rangle| |\langle Z\rangle_\phi - \langle Z\rangle_\psi| \leq \sigma_\phi(Z) + \sigma_\psi(Z)$.

References

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.

³Formal declarations: `MPSTensor.IsNormalTensor.exists_normalGauge`, `MPSTensor.exists_normalGauge_of_isNormal`, `MPSTensor.eventually_one_half_lt_infidelity_of_isNormalTensor` and `MPSTensor.eventually_one_half_lt_infidelity_of_isNormal` in `TNLean/MPS/Preparation/NormalGauge` and `TNLean/MPS/Preparation/DepthLowerBoundNormal`.