

The Constant in the Depth Lower Bound

2026-10-03

Verdict: local-correction (resolved).

1 What the source states

Let $|\phi_N\rangle$ be the normalized translation-invariant matrix product states on N sites generated by a normal tensor A with correlation length $\xi = -1/\ln|\lambda_2| > 0$, and let $|\psi_N\rangle$ be obtained from product states by local circuits of depth T . Theorem 1 of [MSWC24] states that if $T = o(\log N)$, then $\varepsilon(\phi_N, \psi_N) = 1 - |\langle\phi_N|\psi_N\rangle| > 1/2$ for all large N . Its proof is quantitative.¹ With $\mu = c_1 e^{-2T/\xi}$, $\delta > \mu^2/9$ and $k = \lfloor N/(4q) \rfloor$ blocks of length $q = \lceil 2(1+\eta)\xi \ln N \rceil$, it asserts that $k > 2/\delta$ gives $(1-\delta)^{k/2} < 1/e$ “and thus $d(\phi, \psi) > \sqrt{3/4}$ ”, and closes with the chain

$$k > \frac{N}{5q} > \frac{N\gamma}{10\xi \log N} > \frac{18}{c_1^2} e^{4T/\xi} > \frac{2}{\delta}. \quad (1)$$

2 What fails

Two constants in this passage are wrong; neither affects the rate $4/\xi$.

1. From $(1-\delta)^{k/2} < 1/e$ the bound $d(\phi, \psi) > 1 - (1-\delta)^{k/2} - \sqrt{2c_0}N^{-\eta}$ gives $d(\phi, \psi) > 1 - 1/e - o(1) \approx 0.632$, not $d > \sqrt{3/4} \approx 0.866$. The conclusion $\varepsilon > 1/2$ needs $\sqrt{3/4}$, because $d(\phi, \psi) = \sqrt{1 - |\langle\phi|\psi\rangle|^2}$ for pure states. It is reached once $(1-\delta)^{k/2} < 1 - \sqrt{3/4} - o(1)$, for example once $k > 5/\delta$, since $2\ln(1/(1 - \sqrt{3/4})) < 4.1$.
2. The factor γ in (1) is not defined in the proof. With $q = \lceil 2(1+\eta)\xi \ln N \rceil$ one has $N/(5q) \geq N/(10(1+\eta)\xi \ln N)$ up to the rounding of q , so γ stands for a constant close to $1/(1+\eta)$.

The factor $\frac{1}{2}$ missing from the trace-distance bound, recorded in `docs/paper-gaps/mswc24_decaying_correlations_windowed_connected.tex`, also

¹Source traceability: section “Proof of Theorem 1” of the Supplemental Material, the last two paragraphs of the proof; source at <https://arxiv.org/abs/2307.01696>.

changes only c_1 . After these repairs the chain still reads $N/\ln N \geq K e^{4T/\xi}$ with K depending only on A ; so $\varepsilon(\phi_N, \psi_N) \leq 1/2$ forces

$$T \geq \frac{\xi}{4}(\ln N - \ln \ln N) - C_A. \quad (2)$$

The source's argument does not give $T \geq (\xi/4) \ln N - C_A$ without the correction $\ln \ln N$, which comes from the block length $q \propto \ln N$.

3 What is formalized

The formal bound is derived from the averaging argument of the blueprint, which replaces the source's fidelity of blocked product states (see `docs/paper-gaps/mswc24_depth_lower_bound_gauge.tex`), and carries the same rate. For a normal tensor in the gauge of the source's eq. (5), a unit vector ψ prepared in depth T with $|\langle \phi_N | \psi \rangle| \geq 1/2$ and $N \geq B(T+1)$ satisfies

$$N \leq C(T+1)e^{4T/\xi}, \quad (3)$$

with B, C independent of N and T . The exponent is the square of the correlator bound $c e^{-(s'-1)/\xi}$ at a separation $s' \leq 2T + O(1)$, and the factor $T+1$ comes from the spacing of the windows, as the factor $\ln N$ in (1) comes from the block length. Taking logarithms, $\varepsilon(\phi_N, \psi) \leq 1/2$ forces $(\xi/4) \ln N \leq T + (\xi/4) \ln(T+1) + C'$ for every normal tensor, with ξ computed from λ_2/λ_1 ; since $T < (\xi/4) \ln N$ otherwise, this implies (2). For every $a < \xi/4$ it gives $T \geq a \ln N - C_a$, and with the source's eq. (1) the optimal depth at fixed error $0 < \varepsilon \leq 1/2$ is $\Theta(\log N)$.²

4 Verdict

The two slips are in constants of the closing inequality; the theorem and the rate $4/\xi$, that is the constant $\xi/4$ in front of $\ln N$, stand. The formal statements carry the source's rate with the logarithmic correction written as $(\xi/4) \ln(T+1)$.

References

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.

²Formal declarations: `MPSTensor.exists_depth_lower_bound` in `TNLean/MPS/Preparation/DepthLowerBound`; `MPSTensor.exists_log_le_of_infidelity_le_one_half`, `MPSTensor.exists_log_le_of_infidelity_le_one_half_of_isNormal`, `MPSTensor.exists_mul_log_le_of_infidelity_le_one_half_of_isNormal` and `MPSTensor.exists_depth_isTheta_log` in `TNLean/MPS/Preparation/DepthLowerBoundConstant`.