

Decaying Correlations: Windows of Separations and the Connected Correlator

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Verdict: false-source (resolved).

1 What the source states

Let A be an injective tensor with transfer map $\mathcal{E}_A$, in the gauge $\sum_i A^{i\dagger} A^i = \mathbb{I}$, $\mathcal{E}_A(\rho) = \rho$, $\rho > 0$, $\text{Tr } \rho = 1$. Let λ_2 be an eigenvalue of $\mathcal{E}_A$ of largest modulus other than 1, with correlation length $\xi = -1/\log|\lambda_2| > 0$, and let $|\phi_N\rangle$ be the normalized translation-invariant matrix product state on N sites. Lemma 2 of the Supplemental Material of [MSWC24] asserts observables $\mathcal{O}, \mathcal{O}'$ of norm one such that for every integer $s > 1$ and all sufficiently large N

$$\langle \phi_N | \mathcal{O}_1 | \phi_N \rangle = \langle \phi_N | \mathcal{O}'_s | \phi_N \rangle = 0, \quad (1)$$

$$\langle \phi_N | \mathcal{O}_1 \mathcal{O}'_s | \phi_N \rangle \geq c e^{-(s-1)/\xi}, \quad (2)$$

with $c > 0$ independent of s and N .¹

2 What holds and what fails

The source’s observables. Let $\ell_2 \neq 0$ be a linear functional with $\ell_2 \circ \mathcal{E}_A = \lambda_2 \ell_2$. The proof of Lemma 2 expands the transfer powers in ordinary eigenvectors, and therefore assumes that $\mathcal{E}_A$ is diagonalizable. Under that assumption it chooses, by injectivity, observables whose one-site maps are the rank-one maps

$$\mathcal{E}_{\mathcal{O}}(Z) = \ell_2(Z) \rho, \quad \mathcal{E}_{\mathcal{O}'}(Z) = \text{Tr}(Z) R_2, \quad (3)$$

with R_2 an eigenvector for λ_2 normalized so that $\ell_2(R_2) = 1$, rescaled to norm one. If λ_2 has a nontrivial Jordan block, every left eigenvector of that block vanishes on every right eigenvector of the same block. The repair recorded below keeps ℓ_2 a left eigenvector and takes R_2 in the generalized eigenspace

¹Source traceability: label `1m:2` of the arXiv LaTeX source, section “Proof of Theorem 1” of the Supplemental Material; the two displays are labelled `auxform1` and `auxform2` there. No copy of this source is kept in `Papers/`.

with $\ell_2(R_2) = 1$. Since ℓ_2 is a left eigenvector for λ_2 and ρ an eigenvector for $1 \neq \lambda_2$, $\ell_2(\rho) = 0$; since trace preservation makes Tr a left eigenvector for 1, it vanishes on the generalized eigenspace of λ_2 , so $\text{Tr } R_2 = 0$. Both one-point functions are therefore 0 at every $N \geq s$, the chain lengths on which the site s exists, so (1) is exact and algebraic, not asymptotic. Because $\ell_2 \circ \mathcal{E}_A^{s-2} = \lambda_2^{s-2} \ell_2$, the two-point function is exactly

$$\langle \phi_N | \mathcal{O}_1 \mathcal{O}'_s | \phi_N \rangle = \frac{\lambda_2^{s-2}}{\text{Tr } \mathcal{E}_A^N}, \quad (4)$$

up to the positive constant of the rescaling, and $\text{Tr } \mathcal{E}_A^N \leq D^2$. Hence the modulus form

$$|\langle \phi_N | \mathcal{O}_1 \mathcal{O}'_s | \phi_N \rangle| \geq c e^{-(s-1)/\xi} \quad (5)$$

holds for every $s \geq 2$ and every $N \geq s$.

The printed real inequality. When λ_2 is not real, the pair (3) cannot be Hermitian: a Hermitian observable has a map with $\mathcal{E}_{\mathcal{O}}(Z^\dagger) = \mathcal{E}_{\mathcal{O}}(Z)^\dagger$, whereas $R_2^\dagger$ is a generalized eigenvector for $\bar{\lambda}_2 \neq \lambda_2$, so $\ell_2(R_2^\dagger) = 0 \neq \overline{\ell_2(R_2)}$. The correlator (4) is then complex, and the printed inequality (2) is false: writing $\lambda_2 = |\lambda_2|e^{i\theta}$, the real part is a multiple of $|\lambda_2|^{s-2} \cos((s-2)\theta)$, which is negative for infinitely many s since θ is not a multiple of π ; it vanishes at some s only for particular θ , such as $\theta = \pi/2$, and never for $\theta = \pi/3$.

The printed inequality also fails when λ_2 is real and negative. Then $\mathcal{E}_A$ preserves Hermitian matrices, so R_2 and ℓ_2 may be taken Hermitian and the source's rank-one maps are realized by Hermitian observables, but the correlator (4) has the sign of λ_2^{s-2} and is negative at every odd s . For example, the qubit tensor with Kraus matrices $\sqrt{q_0} I$, $\sqrt{q_1} \sigma_x$, $\sqrt{q_2} \sigma_y$, $\sqrt{q_3} \sigma_z$, where $q_0 = 1/16$ and $q_1 = q_2 = q_3 = 5/16$, is injective, and every nontrivial eigenvalue of its transfer map equals $-1/4$. Non-Hermiticity of the source's pair is therefore forced only for nonreal λ_2 , while the printed inequality fails for every nonreal and every negative λ_2 .

The Hermitian reading. Reading $\mathcal{O}, \mathcal{O}'$ as Hermitian does not rescue the printed form. Put $P(Z) = \text{Tr}(Z)\rho$ and $t = s - L - 1$, where the observables act on L sites ($L = 1$ in the source). For Hermitian observables the limit of the connected correlator as $N \rightarrow \infty$ is

$$G_\infty(\mathcal{O}, \mathcal{O}'; s) = \text{Tr} \left(\mathcal{E}_{\mathcal{O}}((\mathcal{E}_A - P)^t(\mathcal{E}_{\mathcal{O}'}(\rho))) \right), \quad (6)$$

which is real. When λ_2 and $\bar{\lambda}_2$ are the only eigenvalues of modulus $|\lambda_2|$ and both are semisimple, its leading part is

$$|\lambda_2|^t (a e^{it\theta} + \bar{a} e^{-it\theta}) = 2|a| |\lambda_2|^t \cos(t\theta + \varphi), \quad (7)$$

with $a = |a|e^{i\varphi}$, and the remainder is smaller by a factor tending to zero in t . When θ/π is irrational the cosine comes arbitrarily close to zero for infinitely

many t , so for every Hermitian pair and every $c > 0$ the correlator falls below $c e^{-(s-1)/\xi}$ for infinitely many s . An explicit injective example has $d = 4$, $D = 2$, and a gate angle ψ ,

$$A^0 = \frac{1}{2}\mathbb{1}, \quad A^1 = \frac{1}{2}\text{diag}(1, e^{i\psi}), \quad A^2 = \frac{1}{2}\sigma_x, \quad A^3 = \frac{1}{2}\sigma_y, \quad (8)$$

with $\sum_i A^{i\dagger} A^i = \mathbb{1}$ and transfer spectrum $\{1, (1+e^{i\psi})/4, (1+e^{-i\psi})/4, 0\}$. Since $1+e^{i\psi} = 2\cos(\psi/2)e^{i\psi/2}$, for $-\pi < \psi < \pi$ the phase $\theta = \arg(\lambda_2)$ of (7) is $\psi/2$. At $\psi = \pi/2$ one has $\theta = \pi/4$, so $t\theta$ is an odd multiple of $\pi/2$ at $t = 2, 6, 10, \dots$, and the Hermitian parts of the source's observables (3) have one-point functions 0 and a correlator that vanishes exactly at $s = 4, 8, 12, \dots$ for every $N \geq s$ and is negative at $s = 5, 6, 7$.

The rest of the proof. The source's Theorem 1 is true, and its proof goes through with the modulus form (5) for the source's pair (3): the lower bound on the trace distance comes from the duality $|\text{Tr}((\rho_0 - \sigma_1)\mathcal{Q})| \leq \|\rho_0 - \sigma_1\|_1 \|\mathcal{Q}\|$, which needs no Hermiticity, and the minimization that follows uses only moduli. Four further points need repair, none of which changes the conclusion.

1. Lemma 2 is stated for injective A but applied in Theorem 1 to a normal A on unblocked sites. For normal A the map $\mathcal{Q} \mapsto \mathcal{E}_{\mathcal{Q}}$ is onto only for observables on L sites, where products of L matrices span the matrix algebra.
2. The proof expands $\mathcal{E}_A^{s-1} = \sum_i \lambda_i^{s-1} |R_i\rangle\langle L_i|$ and so assumes $\mathcal{E}_A$ diagonalizable. If λ_2 has a nontrivial Jordan block, a left eigenvector and a right eigenvector of that block pair to zero. The repair takes ℓ_2 a left eigenvector and R_2 in the generalized eigenspace with $\ell_2(R_2) = 1$; then $\ell_2(\mathcal{E}_A^t R_2) = \lambda_2^t$ and still $\text{Tr } R_2 = 0$.
3. The trace-distance bound (display `auxbound`) omits a factor $\frac{1}{2}$: the duality above gives $d(\rho_0, \sigma_1) \geq \frac{1}{2} |\tilde{a}_{\mathcal{Q}} - b_{\mathcal{Q}}|$.
4. The choice $s = 2T + 1$ places the observables at distance exactly $2T$, whereas the source's factorization statement covers distances larger than $2T$; the choice is valid only for a circuit with a sharp light cone. The formal statements require distance larger than $2T$.

3 The corrected statement

Let A be normal, with products of $L \geq 1$ matrices spanning the matrix algebra, in the same gauge, and put

$$G_N(X, Y; s) = \langle \phi_N | X_1 Y_s | \phi_N \rangle - \langle \phi_N | X_1 | \phi_N \rangle \langle \phi_N | Y_s | \phi_N \rangle. \quad (9)$$

There are Hermitian $\mathcal{O}, \mathcal{O}'$ of norm one, $c > 0$, $K \in \{1, 2\}$ with $K = 1$ when λ_2 is real, and $s_0 \geq L + 2$ such that every $s \geq s_0$ has some $s' \in \{s, \dots, s + K - 1\}$

with

$$|G_N(\mathcal{O}, \mathcal{O}'; s')| \geq c e^{-(s'-1)/\xi} \quad \text{for all } N \geq 3s'. \quad (10)$$

The window of K consecutive separations removes the zeros of (7): the Vandermonde matrix of the distinct unimodular numbers $e^{\pm i\theta}$ is invertible, so of any two consecutive values of $a e^{it\theta} + \bar{a} e^{-it\theta}$ one has modulus at least a fixed $c_0 > 0$. The statement uses Hermitian observables because the proof of Theorem 1 below controls variances of Hermitian operators; for such observables the window is necessary, by the example (8). The connected correlator makes the bound independent of the one-point functions; it does not correct the source, whose choice (3) makes them vanish exactly. For injective A and $L = 1$ the hypotheses are those of the source.²

4 Verdict

The printed Lemma 2 is false as printed: the qubit tensor above is injective and nondegenerate, satisfies the source’s hypotheses, and violates the printed inequality at every odd s . More generally the inequality fails as a real inequality when λ_2 is not real or is negative, and fails for Hermitian observables in general. The repair is nevertheless local: the modulus form (5) holds for the source’s own observables, and the source’s Theorem 1 is true. The formal statement (10) is the Hermitian variant; its proof is the source’s spectral argument together with the Vandermonde bound, and it suffices for Theorem 1: that proof needs one separation s' of order $T_N + L$ for each N , and moving s' by at most $K - 1 \leq 1$ sites preserves $s' = o(\log N)$ and $N \geq 3s'$.

The proof of Theorem 1 in the blueprint replaces the source’s product-fidelity step by the concentration of averages of translated observables. Tracking its constants, a state prepared in depth T with overlap at least η with ϕ_N requires

$$T \geq \frac{\xi}{4} (\ln N - \ln \ln N) - \xi \ln \frac{1}{\eta} - C_A, \quad (11)$$

with C_A depending only on A . The same constant $\xi/4$ is implicit in the source’s final inequality, which compares $N/\ln N$ with a multiple of $e^{4T/\xi}$; only its explicit statement is new.

References

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.

²Formal declaration: `MPSTensor.exists_decayingCorrelations` in `TNLean/MPS/Preparation/DecayingCorrelationBound`.