

Coherent Canonical Sector Encoders

2026-10-05

Verdict: scope-restriction (resolved).

1 Source and precise interpretation

The discussion and outlook of [MSWC24] proposes logarithmic-depth conversion of matrix product states in the same phase. Equation (25) and the paragraph “Long-range MPS using measurements” provide the simultaneous sector construction relevant to that claim. They do not specify a canonical logical basis of the whole sector span, an operator-norm conversion bound, or an external-reference statement. The operator-norm and all-accuracy results here give a precise whole-encoding enhancement of that discussion, not a formalization of a new phase predicate.

For an explicitly labelled family of b retained sectors, define the actual column matrix and its fixed polar encoder by

$$V_{A,N}e_j = |\phi_N(A_j)\rangle, \quad F_{A,N} = V_{A,N}(V_{A,N}^\dagger V_{A,N})^{-1/2}. \quad (1)$$

The inverse is taken on the support. The endpoint depends on the family and ring length, and is independent of the requested accuracy or the partition used to approximate it. Its range is exactly the original periodic-sector span, and independent columns make it an isometry.

2 Proved finite-length bounds

The QIClean companion owns the generic estimates. If W is an isometry, $\eta = \|V^\dagger V - I\|$ and $\zeta = \|W^\dagger V - I\|$, then $\|V - W\|^2 \leq \eta + 2\zeta$. Writing F for the polar isometry of V , the square-root bound at the identity and the polar factorization give

$$\|F - W\| \leq \sqrt{\eta + 2\zeta} + \eta \quad (2)$$

for injective V . The cross matrix is complex; replacing its entries by absolute overlaps would discard the relative logical phases.

One register encoding J is an isometry on the logical space. The existing simultaneous block-unitary construction gives $U_X J = W_X$ as a whole matrix,

with W_X isometric and depth at most a constant times the maximum block length. The same chosen circuit acts on all logical amplitudes. For two such physical unitaries, $U = U_B U_A^\dagger$ has depth $T_A + T_B$ and

$$\|U F_{A,N} - F_{B,N}\| \leq \|F_{A,N} - W_A\| + \|F_{B,N} - W_B\|. \quad (3)$$

There is no cost for producing the common seed from a product state: it cancels algebraically. A measurement protocol is never inverted.

For every finite reference space and every input ξ , the same operator bound δ yields a vector error at most $\delta\|\xi\|$ after tensoring with the reference identity. The proof sums squared norms of reference slices, with no reference-dimension factor. No channel diamond-norm theorem is asserted by this statement.

3 Proved exact construction through a fixed virtual space

For injective virtual placements with disjoint ranges, let $B = \bigoplus_j A_j$ be the unweighted direct sum of bond dimension D . Let C_N be its blocked physical matrix on configurations, S_N its polar partial isometry, $P_N = (C_N^\dagger C_N)^{1/2}$, and T the sector-trace selector. The identity $C_N T = V_{A,N}$ gives, with $R_N = P_N T$,

$$V_{A,N} = S_N R_N, \quad F_{A,N} = S_N Q_N, \quad (4)$$

where Q_N is the polar partial isometry of R_N . The intermediate space is $\mathbb{C}^{D^2}$, with dimension independent of the ring length.

If the blocked sum is injective on its sector-pair support and all sector bond dimensions are positive, the trace selectors show directly that $V_{A,N}$ is injective. Then Q_N is an isometry supported on that same virtual set. Its columns, with the one-block cyclic pair orientation, form an orthonormal family to which the simultaneous genuine-unitary compiler applies. For fixed encodings into registers of width $r \geq 2$, a constant C is chosen before the tensors and N . At any supported- injective length $N \geq 3r$, one circuit satisfies $U_N J_N = F_{A,N}$ exactly and has depth at most CN . Pair support is needed only at the actual single block. The whole logical encoder is transported; the construction uses fixed-width pair synthesis and sequential blocked-polar unitaries.

4 Proved uniform logarithmic-depth conversion

Fix two explicitly matched families of $b > 0$ distinct normal sectors with common physical dimension $d \geq 2$. Each sector is left canonical and has a faithful stationary density of trace one; mixed-transfer spectral radii within each family are strictly less than one. The formal theorem chooses $C > 0$ and $N_0 \geq 2$ before the ring length and accuracy. For every $N \geq N_0$, the fixed encoders are isometries, and for every $\varepsilon \in (0, 1]$ one genuine nearest-neighbor circuit U

satisfies

$$\text{depth}(U) \leq C \log(N/\varepsilon), \quad (5)$$

$$\|UF_{A,N} - F_{B,N}\| \leq \varepsilon. \quad (6)$$

The same chosen U satisfies, for every finite reference space $\mathcal{R}$ and every logical-reference vector ξ ,

$$\|[(UF_{A,N} - F_{B,N}) \otimes I_{\mathcal{R}}]\xi\| \leq \varepsilon \|\xi\|. \quad (7)$$

The endpoints remain independent of accuracy and of the selected partition.¹

The proof first obtains Gram and complex cross-matrix estimates with sums over the fixed logical dimension. One common decay rate, register width and eventual support-injectivity cutoff serve both families. If the common block scale q fits inside the ring, one partition has block lengths between q and $2q$. The two genuine encoders share its logical seed, so their relative unitary has the sum of their depths and endpoint errors. The choice $q = \lceil (2/r) \log(N/\varepsilon) + c_0 \rceil$ accounts for the square root in the endpoint estimate. If $q > N$, the exact fixed-width whole-encoder construction above has depth $O(N) \leq O(q)$ and gives the same logarithmic bound. No product-state preparation of the logical seed is charged, and no measurement protocol is inverted.

5 Parent kernels at a valid interaction range

For supplied canonical data of a tensor with physical and bond dimensions $d, D \geq 1$, the encoder of its retained normal representatives has range exactly equal to the kernel of the range- L parent Hamiltonian whenever $3D^5 \leq L \leq N$, including $N = L$.² This is an identity of the actual periodic-sector span. Applying the conversion theorem to such representatives still requires its explicit canonical hypotheses and a matching of the labels; no additional gauge substitution or column rescaling is implicit.

6 Chosen vectors and boundaries

Transporting a fixed logical basis does not equate the normalized periodic amplitudes of two tensors. The biased-versus-balanced ferromagnetic example and its chosen-vector obstruction in [con26] remain applicable. No symmetry-preserving conversion, arbitrary-small- N claim, or implicit transport through a canonical-gauge normalization is asserted here. The physical gate model consists of genuine nearest-neighbor unitaries with unrestricted gate parameters.

¹Formal declaration: `MPSPreparation.exists_log_depth_coherent_sectorEncoder_con`
`version`.

²Formal declaration: `MPSTensor.CPSVCanonicalFormData.range_sectorEncoder_eq_ker`
`_parentHamiltonian`.

References

- [con26] TNLean contributors. Compatible coherent seeds and non-normal state conversion. Technical report, TNLean project, 2026. https://sirui-lu.com/TNLean/paper-gaps/mswc24_same_phase_compatible_seed.pdf.
- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.