

The Positive Part of a Blocked Tensor with Overlapping Blocks

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Verdict: false-source (wip).

1 What the source states

Let $A^i = \bigoplus_{j=1}^b \text{diag}(\mu_{j,1}, \dots, \mu_{j,m_j}) \otimes A_j^i$ with normal A_j forming a basis of normal tensors [MSWC24, eq. (S2)], block q sites, and write the polar decomposition of the blocked map as $B = VP$. The source argues “Since V is an isometry, P inherits the block structure of A ”,

$$P = \bigoplus_{j=1}^b \text{diag}(\mu_{j,1}^q, \dots, \mu_{j,m_j}^q) \otimes P_j \quad (1)$$

with normal P_j [MSWC24, eq. (S5)]. It then approximates the state on $N = qM$ sites by $|\tilde{\phi}_N\rangle \propto V^{\otimes M} \sum_j \beta_j |\Omega_j\rangle$ with $\beta_j = \sum_k \mu_{j,k}^N$ [MSWC24, eq. (S7)], and part (ii) of the approximation-error lemma bounds $\varepsilon = 1 - |\langle \tilde{\phi}_N | \phi_N \rangle|$ by $O((N/q)e^{-\gamma q/\xi_{\text{diag}}})$ for $q = o(N)$ [MSWC24, eq. (S12)], where $\xi_{\text{diag}} = \max_j \xi_{jj}$ is the largest correlation length of the blocks. In the proof, the overlaps of distinct blocks enter only through a term $O(e^{-N/\xi_{\text{off-diag}}})$, which $q = o(N)$ removes.

2 Where the argument fails

The positive part is $P = (B^\dagger B)^{1/2}$, and $B^\dagger B$ is the transfer matrix of B with its legs regrouped. Its block between two distinct blocks A_j and $A_{j'}$ is the q -th power of the mixed transfer matrix $E_{jj'} = \sum_i A_j^{i*} \otimes A_{j'}^i$. The source uses that $E_{jj'}$ has spectral radius $\tau < 1$ [MSWC24, proof of Lemma 1’], but $E_{jj'}^q$ is not zero, so $B^\dagger B$ and its square root are not block diagonal. The block form (1) holds only when the q -site states of distinct blocks are orthogonal. This failure is separate from the one for repeated blocks [con26b]: it occurs when every multiplicity is one.

3 The counterexample

Take $d = 2$ and the two one-dimensional normal blocks $A_1 = (1, 0)$ and $A_2 = (3/5, 4/5)$, each of multiplicity one with weight one, so that $A^0 = \text{diag}(1, 3/5)$ and $A^1 = \text{diag}(0, 4/5)$. Both blocks satisfy $\sum_i |a_i|^2 = 1$ and are not proportional, so they form a basis of normal tensors, and $\beta = (1, 1)$. The q -site states of the blocks are $|0 \cdots 0\rangle$ and $(\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle)^{\otimes q}$, with overlap $s = (3/5)^q$. On the diagonal bond coordinates $(0, 0)$ and $(1, 1)$, the only ones on which B does not vanish,

$$B^\dagger B = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}, \quad P = \begin{pmatrix} u & v \\ v & u \end{pmatrix}, \quad u = \frac{\sqrt{1+s} + \sqrt{1-s}}{2}, \quad v = \frac{\sqrt{1+s} - \sqrt{1-s}}{2}, \quad (2)$$

and the support projector of P is the projector onto these two coordinates. The fixed point of a one-dimensional block is $\sigma = 1$, so the pairs are $\omega_1 = e_{00}$ and $\omega_2 = e_{11}$. Since $V^\dagger B = P$ and $V^\dagger V$ fixes the pairs, for every $q, M \geq 1$,

$$\langle \tilde{\phi}_N | \phi_N \rangle = \frac{u^M + v^M}{(1 + s^M)^{1/2}}. \quad (3)$$

For $q = 1$ this is $(3^M + 1)/(10^M + 6^M)^{1/2}$, which tends to zero. Since $u^2 + v^2 = 1$, $u^2 = (1 + \sqrt{1-s^2})/2 \geq 1/2$, and $1 - u \geq s^2/8$, one has $u^M + v^M \leq u^3 + v^2 = 1 - u^2(1 - u) \leq 1 - s^2/16$ for $M \geq 3$, so

$$\varepsilon \geq \frac{s^2}{16} = \frac{1}{16} \left(\frac{9}{25} \right)^q \quad (M \geq 3). \quad (4)$$

The construction nevertheless converges. For $M = 1$ the overlap (3) is one, since $u + v = \sqrt{1+s}$. For $M \geq 2$, $u \geq u^2 = (1 + \sqrt{1-s^2})/2 \geq 1 - s^2/2$, so $u^M \geq 1 - Ms^2/2$ by Bernoulli's inequality, and $(1 + s^M)^{1/2} \leq 1 + s^2/2$; hence

$$\varepsilon \leq Ms^2 = M \left(\frac{9}{25} \right)^q \quad (q, M \geq 1). \quad (5)$$

The error of this example therefore lies between $s^2/16$ (for $M \geq 3$) and Ms^2 , and it tends to zero whenever $(N/q)\tau^{2q} \rightarrow 0$ with $\tau = 3/5$, for instance when $q \geq c \log N$ with $c > 1/(2 \log(5/3))$.

The transfer maps of one-dimensional blocks are the identity and have no eigenvalue other than one. The source defines $\xi_{jj} = -1/\ln|\lambda_2^{(j)}|$ with $\lambda_2^{(j)}$ the subleading eigenvalue of the transfer matrix of A_j [MSWC24]; here there is none, so ξ_{diag} is degenerate. Under the usual convention $\lambda_2^{(j)} = 0$ it is 0, and the printed bound then asserts $\varepsilon = 0$, which (4) refutes directly. The formal statement avoids the convention: it reads ξ_{diag} as the correlation length at any λ_2 with $0 < |\lambda_2| < 1$ bounding the subleading eigenvalues, which here is every such λ_2 , and covers those with $r = e^{-\gamma/\xi_{\text{diag}}} = |\lambda_2|^\gamma < 9/25$. Take the sequence $q = M$, $N = q^2$, which satisfies $q = o(N)$. The printed bound $C(N/q)e^{-\gamma q/\xi_{\text{diag}}} = Cqr^q$ is eventually smaller than $(9/25)^q/16$ for every constant C , and so is $Cqr^q \exp(Cqr^q)$. The error is governed by the overlap $s = \tau^q$ of the blocks, with

$\tau = 3/5$ the eigenvalue of the mixed transfer matrix: it is at least $s^2/16$ and at most $Ms^2 = (N/q)\tau^{2q}$, and numerically close to $Ms^2/8$, a contribution that the source's estimate does not contain. What fails is the rate, not the construction.

4 The construction converges

Let every block have multiplicity one and weight one, so that $\beta_j = 1$ and $\alpha_j = b^{-1/2}$, and let the blocks A_j be normal in the gauge $\sum_i (A_j^i)^\dagger A_j^i = \mathbf{1}$ with positive definite fixed points σ_j of trace one. Let λ_2 bound the moduli of the eigenvalues other than one of every transfer matrix E_{jj} and the moduli of all eigenvalues of the mixed transfer matrices $E_{jj'}$, $j \neq j'$, so that $\xi = -1/\ln|\lambda_2|$ is at least ξ_{diag} and $\xi_{\text{off-diag}}$, and let $0 < \gamma < 1$. Then there is C with

$$\varepsilon \leq Cy e^{Cy} \quad \text{and} \quad \varepsilon \leq Cy, \quad y = \frac{N}{q} e^{-\gamma q/\xi}, \quad (6)$$

for every $q \geq 1$ and $M \geq 1$, for the source's state (S7) itself: the same V , the same pairs and the same weights. No orthogonality and no condition $q = o(N)$ is needed. This is the statement of part (ii) with ξ_{diag} replaced by $\max(\xi_{\text{diag}}, \xi_{\text{off-diag}})$, for $0 < \gamma < 1$ in place of the source's $0 < \gamma < 1/2$.

The proof replaces the block form (1) by an estimate. With K_j the isometries placing the bond pairs of block j , the blocked map is $B = \sum_j B_j K_j^\dagger$ and $B^\dagger B = \sum_{j,j'} K_j B_j^\dagger B_{j'} K_{j'}^\dagger$. The diagonal blocks satisfy $\|B_j^\dagger B_j - \sigma_j^T \otimes \mathbf{1}\| \leq K e^{-\gamma q/\xi}$ as in the normal case, and the off-diagonal blocks are the entries of $E_{j'j}^q$, so $\|B_j^\dagger B_{j'}\| \leq K e^{-\gamma q/\xi}$. Their common limit $G_\infty = \sum_j K_j (\sigma_j^T \otimes \mathbf{1}) K_j^\dagger$ vanishes on the bond pairs (a, b) with a and b in different blocks, and so does $B^\dagger B$. Adding the projector Q onto these pairs makes $G_\infty + Q$ positive definite, and $PQ = 0$ gives $\sqrt{B^\dagger B + Q} = P + Q$. The square root is Lipschitz at a positive definite point, so $\|P - P_\infty\| \leq K e^{-\gamma q/\xi}$ with $P_\infty = \sum_j K_j ((\sqrt{\sigma_j})^T \otimes \mathbf{1}) K_j^\dagger$.

Since $V^\dagger B = P$, the overlap of the unnormalized state $V^{\otimes M} \sum_j \alpha_j |\Omega_j\rangle$ with $|\phi_N(A)\rangle$ is $b^{-1/2} \sum_j z_j$ with $z_j = \langle \Omega_j | \phi_N(P) \rangle$, and $V^{\otimes M}$ does not increase norms, so normalizing the approximating state only increases the overlap. The mixed transfer matrix of P_∞ against the fixed-point tensor of block j , placed in the full bond space, is $\rho \mapsto \text{Tr}(\Pi_j \rho) \tilde{\sigma}_j$, with Π_j the projector onto the bond coordinates of block j and $\tilde{\sigma}_j$ the embedded fixed point; it is idempotent with trace one. The telescoping estimate of the normal case then gives $|z_j - 1| \leq Cy e^{Cy}$. Finally $\|\phi_N(A)\|^2 = \sum_{j,j'} \langle \phi_N(A_j) | \phi_N(A_{j'}) \rangle$, whose diagonal terms are one up to $O(e^{-\gamma N/\xi})$ and whose other terms are $O(e^{-\gamma N/\xi})$. With $S = \sum_j z_j$ and $t = \|\phi_N(A)\|$, the inequality $1 - |S|/(\sqrt{b}t) \leq |S - b| + |t^2 - b|$ for $b \geq 1$ gives (6).

The first-order Lipschitz estimate gives the rate $e^{-\gamma q/\xi}$; the counterexample and the numerical checks below show the error to be of order $(N/q)\tau^{2q}$, of second order in the overlaps. The next section proves this rate.

5 The second-order rate

Keep the hypotheses of the previous section. Then there is C with

$$\varepsilon \leq Cy e^{Cy} \quad \text{and} \quad \varepsilon \leq Cy, \quad y = \frac{N}{q} e^{-2\gamma q/\xi}, \quad (7)$$

for every $q \geq 1$ and $M \geq 1$, for the source's state (S7). Since ξ may be the correlation length of any λ_2 with $|\lambda_2|$ at least the spectral radius τ of the mixed transfer matrices and the subleading moduli of the blocks, the error is $O((N/q)|\lambda_2|^{2\gamma q})$ for every $\gamma < 1$. For the counterexample, $\lambda_2 = \tau = 3/5$ is admissible, and the error is at least $\tau^{2q}/16$ for $M \geq 3$; so the exponent $2q/\xi$ is sharp up to $\gamma < 1$, and so is the factor M , by the numerical check $\varepsilon \approx M\tau^{2q}/8$.

The positive part converges to P_∞ only to first order in the overlaps: its blocks between distinct blocks are of order τ^q . For the displayed example, the active Gram matrix is $G_q = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} > 0$ for $q \geq 1$, with $s = (3/5)^q$. Its square root has entries $u, v \geq 0$ satisfying $u^2 + v^2 = 1$ and $2uv = s$, so $u \leq 1$ and $v \geq s/2$. Consequently,

$$\|\sqrt{G_q} - I\| \geq s/2, \quad \|P_q - P_\infty\| \geq s/2, \quad (8)$$

where $P_q = J\sqrt{G_q}J^\dagger$ and $P_\infty = JJ^\dagger$. For every $C \geq 0$ and every threshold there is a larger positive q with $Cs^2 < s/2$. Hence even an eventual matrix-norm bound $O(s^2)$ is impossible; the quadratic state error does not follow from a quadratic square-root norm estimate. What is of second order is the dependence of the overlaps z_j on P . As in the normal case, put $\iota(X) = (X^i \sqrt{\tilde{\sigma}_j})_i$ for a tensor X with physical index the output pair. The mixed transfer matrix τ_q of P against the fixed-point tensor $P'_{j,\infty}$ of block j is compressed by the idempotent $R_j : \rho \mapsto \text{Tr}(\Pi_j \rho) \tilde{\sigma}_j$ to $R_j \tau_q R_j = \alpha R_j$, with $\alpha = \text{Tr}(\Pi_j \tau_q(\tilde{\sigma}_j)) = \langle \iota(P'_{j,\infty}) | \iota(P) \rangle$, because $\sqrt{\tilde{\sigma}_j} \Pi_j = \sqrt{\tilde{\sigma}_j}$. Three facts make α close to one to second order.

1. $\tilde{\sigma}_j$ is a fixed point of the transfer map of the direct sum, so $\|\iota(P)\|^2 = \text{Tr } E_A^q(\tilde{\sigma}_j) = 1$, and $\|\iota(P'_{j,\infty})\| = 1$.
2. $\iota(P_\infty) = \iota(P'_{j,\infty})$: the inner product of the two is $\text{Tr } R_j = 1$, and $\iota(P_\infty)$ is a unit vector as the limit of the unit vectors $\iota(P)$. Hence $1 - \text{Re } \alpha = \frac{1}{2} \|\iota(P) - \iota(P_\infty)\|^2 = O(\|P - P_\infty\|^2)$.
3. $\alpha = \text{Tr}(P(\sqrt{\tilde{\sigma}_j}^T \otimes \tilde{\sigma}_j))$ is real, since P and $\tilde{\sigma}_j$ are Hermitian.

Writing $\tau_q = \alpha(R_j + Z)$ with $R_j Z R_j = 0$, the trace of $(R_j + Z)^M$ is one up to $O(M\|Z\|^2)$ for $M \geq 2$, and $0 \leq 1 - \alpha^M \leq M(1 - \alpha)$; the reality of α keeps the phases α^M of the blocks from adding a term of order $M^2\|P - P_\infty\|^2$ to $|\sum_j z_j|$. So $|z_j - 1| \leq Cy e^{Cy}$. The norm of the target satisfies $|\|\phi_N(A)\|^2 - b| = O(e^{-\gamma N/\xi})$, which is $O(e^{-2\gamma q/\xi})$ for $M \geq 2$. With $s = |\sum_j z_j|/\sqrt{b}$ and $t = \|\phi_N(A)\|$, the error is at most $1 - s/t \leq 2(t^2 - s^2)$ once $t^2 \geq 1/2$, and $t^2 - s^2$ is at most $O(y)$. For $M = 1$ the norm of the target is only first-order close to b , and a different estimate is used: with $u = \phi_1(P)$ and $W = \sum_j \phi_1(P'_{j,\infty})$, one

has $\|W\|^2 = b$, $\sum_j z_j = \langle W|u \rangle$, $t^2 - s^2 \leq \|u - W\|^2$, and $W = \phi_1(P_\infty)$, so $t^2 - s^2 = O(\|P - P_\infty\|^2)$.

6 Weights of different moduli

Let every block still have multiplicity one, now with weight μ_j . For multiplicity one the phase of μ_j can be absorbed into A_j , which stays normal with the same transfer map; the weight of block j in (S7) is then $\beta_j = |\mu_j|^N$. So let $0 \leq w_j \leq 1$ be real with $\max_j w_j = 1$, the normalization of the source [MSWC24, after eq. (S2)], let $A = \bigoplus_j w_j A_j$ and $\beta_j = w_j^N$, and keep the hypotheses of the previous section on the blocks, on λ_2 and on γ . Then (6) holds for the source's state (S7), with a constant C that does not depend on q or M , again for every $q \geq 1$ and $M \geq 1$.

The Lipschitz step of the previous section fails: the limit of the Gram matrices is now $G_\infty^{(q)} = \sum_j w_j^{2q} K_j (\sigma_j^T \otimes \mathbf{1}) K_j^\dagger$, whose smallest eigenvalue on the blocks of weight $w_j < 1$ tends to zero as q grows. Three changes repair the argument.

1. Writing $B = \sum_j w_j^q B_j K_j^\dagger$, every block of $B^\dagger B - G_\infty^{(q)}$ is a block of the unit-weight case multiplied by $w_j^q w_{j'}^q \leq 1$, so $\|B^\dagger B - G_\infty^{(q)}\| \leq K e^{-\gamma' q / \xi}$ for every $0 < \gamma' < 1$, with K independent of the weights.
2. The square root is $\frac{1}{2}$ -Hölder on positive semidefinite matrices, $\|\sqrt{X} - \sqrt{Y}\| \leq \|X - Y\|^{1/2}$, the bound used in [PSC21, Supplemental Material, eq. (26)]. With $\gamma' = 2\gamma < 1$ it gives $\|P - P_\infty^{(q)}\| \leq K^{1/2} e^{-\gamma q / \xi}$, where $P_\infty^{(q)} = \sum_j w_j^q K_j ((\sqrt{\sigma_j})^T \otimes \mathbf{1}) K_j^\dagger$. The halving of the exponent is absorbed by the range $0 < \gamma < 1/2$ of the source.
3. The mixed transfer matrix of $P_\infty^{(q)}$ against the fixed-point tensor of block j is w_j^q times the rank-one idempotent $R_j : \rho \mapsto \text{Tr}(\Pi_j \rho) \tilde{\sigma}_j$ of trace one. Its powers $w_j^{qk} R_j$ are bounded by $\|R_j\|$, uniformly in q , so the telescoping estimate gives $|z_j - w_j^N| \leq C_j y e^{C_j y}$.

With $S = \sum_j \beta_j z_j$ and $b = \sum_j \beta_j^2 \geq 1$ one has $|S - b| \leq \sum_j |z_j - \beta_j|$, and as before $\|\phi_N(A)\|^2 - b \leq K e^{-\gamma N / \xi}$; the inequality of the previous section gives (6).

The Hölder step limits this argument to $0 < \gamma < 1/2$. The bound (6) holds for every $0 < \gamma < 1$, with a constant that does not depend on the weights: blocks of weight zero contribute nothing and may be dropped, and for nonzero weights the direct sum is the case of multiplicity one of the repeated blocks of [con26c], where the positive part is estimated relative to the weights and no exponent is lost.

The phase cannot be kept in β_j . Write $\mu_j = |\mu_j| u_j$ with $u_j = e^{i\theta_j}$, let B_0 be the blocked map for the weights $|\mu_j|$, and let U be the diagonal unitary that multiplies the bond coordinates of block j by u_j^q . Then $B = B_0 U$, so

$B^\dagger B = U^\dagger B_0^\dagger B_0 U$, $P = U^\dagger P_0 U$ and $V = V_0 U$. The diagonal blocks of P are those of P_0 and carry no phase, contrary to (1); only the mixed blocks of P , which vanish for orthogonal blocks, carry the phases $\bar{u}_j^q u_j^q$. Since $|\Omega_j\rangle$ lies on the bond coordinates of block j , $V^{\otimes M}|\Omega_j\rangle = e^{iN\theta_j} V_0^{\otimes M}|\Omega_j\rangle$, and with $\beta_j = \mu_j^N$ the state (S7) has the block coefficients $|\mu_j|^N e^{2iN\theta_j}$ relative to $V_0^{\otimes M}|\Omega_j\rangle$, in which the phase of the weight appears twice. For the random blocks of the numerical check below with $\mu = (1, e^{0.9i})$ and $M = 8$, the error with $\beta_j = \mu_j^N$ is 0.794, 0.191, 0.301 at $q = 4, 8, 12$, against 0.355 , $6.53 \cdot 10^{-2}$, $1.47 \cdot 10^{-2}$ with $\beta_j = |\mu_j|^N$. The printed form $\beta_j = \mu_j^N$ is consistent with the block form (1), which assigns the phase to the diagonal blocks of P . With the printed coefficients (S6) and (S7) fail already for orthogonal blocks of multiplicity one (Section 7), and $\beta_j = |\mu_j|^N$ is used, with the phases absorbed into the blocks. The coefficients $\beta_j = \mu_j^N$ of (S4) remain correct in the expansion (S3) of $|\phi_N\rangle$.

7 The phase of a weight: a counterexample

The doubled phase is a separate failure of (S7), which occurs already for orthogonal blocks of multiplicity one. Take $d = 2$, the one-dimensional normal blocks $A_1 = (1, 0)$ and $A_2 = (0, 1)$, each of multiplicity one, with weights 1 and μ , $|\mu| = 1$, so that $A^0 = \text{diag}(1, 0)$ and $A^1 = \text{diag}(0, \mu)$ and $|\phi_N\rangle \propto |0 \cdots 0\rangle + \mu^N |1 \cdots 1\rangle$. The blocks form a basis of normal tensors in canonical form II, and both weights have modulus one. For $q \geq 1$ the Gram matrix of the blocked map on the diagonal bond coordinates is the identity, so $P = JJ^\dagger$ and $V = BJJ^\dagger$, with J the isometry placing the coordinate e on the pair (e, e) ; V maps the pair of the second block to $\mu^q |1 \cdots 1\rangle$. For coefficients α with $|\alpha_1|^2 + |\alpha_2|^2 = 1$ the overlap is $(\bar{\alpha}_1 + \bar{\alpha}_2)/\sqrt{2}$, so for every q , $M \geq 1$

$$\langle \tilde{\phi}_N | \phi_N \rangle = \frac{1 + \bar{\mu}^N}{2} \quad (\beta = (1, \mu^N)), \quad \langle \tilde{\phi}_N | \phi_N \rangle = 1 \quad (\beta = (1, |\mu|^N)). \quad (9)$$

For $\mu = -1$ and odd N the state (S7) is $\propto |0 \cdots 0\rangle + |1 \cdots 1\rangle$, orthogonal to $|\phi_N\rangle \propto |0 \cdots 0\rangle - |1 \cdots 1\rangle$, and $\varepsilon = 1$. The transfer maps of the blocks have no eigenvalue other than one, so every λ_2 with $0 < |\lambda_2| < 1$ bounds their subleading eigenvalues, and along $q = M$ odd, $N = q^2$, the printed bound $C(N/q)e^{-\gamma q/\xi_{\text{diag}}} \exp(C(N/q)e^{-\gamma q/\xi_{\text{diag}}})$ tends to zero while $\varepsilon = 1$. A script evaluating the overlap from the eigendecomposition of $B^\dagger B$ reproduces (9) for $\mu \in \{-1, i, e^{0.9i}\}$ and $(q, M) \in \{(1, 1), (1, 3), (2, 3), (3, 3), (3, 5), (5, 5)\}$.

The same tensor is the direct sum $A_1 \oplus A'_2$ with the block $A'_2 = (0, \mu)$ and weights $(1, 1)$, for which (S4) gives $\beta = (1, 1)$ and (S7) is exact: the printed coefficients depend on how the phase is split between the block and the weight. The coefficients $\beta_j = |\mu_j|^N$ do not.

8 What is formalized

The overlap (3) for all $q, M \geq 1$, the lower bound (4), the upper bound (5), the fact that every λ_2 bounds the subleading eigenvalues of the two blocks, and the failure of the printed bound along $q = M$ are formalized.¹ The positive part and the support projector are computed from the uniqueness of the positive square root of $B^\dagger B$. The positive definiteness of the active Gram matrix, both lower bounds in (8), and the failure of every eventual quadratic polar-norm bound are also formalized.²

The hypotheses of part (ii) of the lemma are also formalized for this tensor. After ordering its bond coordinates it is the direct sum $\bigoplus_j \mu_{j,1} A_j$ of (S2) for a basis of normal tensors: both blocks are normal, left-canonical with the diagonal positive-definite fixed point 1 (canonical form II), have self-overlap one, give linearly independent states for every $N \geq 1$, are not related by a gauge transformation and a phase, and give states that become orthogonal as $N \rightarrow \infty$; both weights equal one. One theorem collects these facts, with canonical form II stated for the diagonal positive-definite fixed point 1 of each block, the same fixed point from which the approximating state in its conclusion is built; it also collects the identification of the pairs and of the weights β_j with those of the canonical form, and the failure of the printed bound.³

The corrected bound (6) for blocks of multiplicity one and weight one, with its ingredients, is formalized.⁴

The bound (7) at the second-order rate, for blocks of multiplicity one and weight one and every $0 < \gamma < 1$, with its ingredients, is formalized.⁵

¹Formal declarations: `MPSTensor.nonNormalApproxOverlap_overlappingBlockTensor`, `MPSTensor.le_one_sub_norm_nonNormalApproxOverlap_overlappingBlockTensor`, `MPSTensor.one_sub_norm_nonNormalApproxOverlap_overlappingBlockTensor_le`, `MPSTensor.norm_le_of_hasEigenvalue_transferMap_overlappingBlockBasis`, `MPSTensor.not_approximationError_le_overlappingBlockTensor` in `TNLean/MPS/Preparation/OverlappingBlockCounterexample`.

²Formal declarations: `MPSTensor.posDef_diagGram_overlappingBlockDiag`, `MPSTensor.overlappingBlockOverlap_div_two_le_norm_sqrt_diagGram_sub_one`, `MPSTensor.overlappingBlockOverlap_div_two_le_norm_polarPos_sub_limit`, and `MPSTensor.exists_norm_polarPos_overlappingBlockTensor_sub_limit_gt_sq` in `TNLean/MPS/Preparation/OverlappingBlockCounterexample`.

³Formal declarations: `MPSTensor.isBNTCanonicalForm_and_not_approximationError_le_overlappingBlock`, `MPSTensor.isBNTCanonicalForm_overlappingBlockSector`, `MPSTensor.reindex_toTensor_overlappingBlockSector` and `MPSTensor.embeddedFixedPointPair_overlappingBlockSector` in `TNLean/MPS/Preparation/OverlappingBlockCounterexample`.

⁴Formal declarations: `MPSTensor.exists_approximationError_le_overlappingBlockSum` and `MPSTensor.exists_approximationError_le_mul_overlappingBlockSum` in `TNLean/MPS/Preparation/BlockSumError`; `MPSTensor.exists_norm_polarPos_blockTensor_blockSum_sub_le` in `TNLean/MPS/Preparation/OverlappingBlockGram`; `MPSTensor.mixedMapLM_blockSumPosLimit_apply`, `MPSTensor.exists_norm_mpvOverlap_polarPosTensor_blockSum_sub_one_le` and `MPSTensor.exists_abs_norm_mpvState_blockSum_sq_sub_le` in `TNLean/MPS/Preparation/OverlappingBlockOverlap`.

⁵Formal declarations: `MPSTensor.exists_approximationError_le_overlappingBlockSum_sq`, `MPSTensor.exists_approximationError_le_mul_overlappingBlockSum_sq`, `MPSTensor.exists_norm_mpvState_sq_sub_norm_sum_mpvOverlap_sq_le` and `MPSTensor.one_sub_norm_nonNormalApproxOverlap_blockSum_le` in `TNLean/MPS/Preparation/SecondOrderBlockError`; `MPSTensor.exists_abs_norm_mpvState_blockSum_sq_sub_le` in

The bound (6) for real weights $w_j \geq 0$, not all zero, and multiplicity one, for every $0 < \gamma < 1$ and with a constant independent of the weights, is formalized as a case of complex weights below; the case of weight one is derived from it.⁶ The counterexample (9), with the hypotheses of part (ii) for it and the failure of the printed bound for $\mu = -1$, is formalized.⁷ The bound (6) for complex weights μ_j , not all zero, multiplicity one, $\beta_j = |\mu_j|^N$ and every $0 < \gamma < 1$, with a constant independent of the weights, is formalized. Blocks of weight zero are dropped, and for nonzero weights the direct sum is the case of one copy of each block of [con26a], where the coefficients for the pairs on the one copy are $|\mu_j|^N$.⁸

9 Numerical check

A script evaluates the overlap through P , using $V^\dagger B = P$ and the support projector of P , from the eigendecomposition of $B^\dagger B$.

- For the counterexample it reproduces (3), and ε is $3.324 \cdot 10^{-2}$, $4.348 \cdot 10^{-3}$, $5.641 \cdot 10^{-4}$ at $M = 16$ and $q = 4, 6, 8$, against $Ms^2/8 = 3.359 \cdot 10^{-2}$, $4.354 \cdot 10^{-3}$, $5.642 \cdot 10^{-4}$.
- For two random normal blocks of bond dimension two on $d = 2$ with subleading eigenvalues 0.4312 and 0.4481 and mixed spectral radius $\tau = 0.780$, with unit weights, $\varepsilon/(M\tau^{2q})$ is 0.176, 0.167, 0.154, 0.155 at $M = 8$ and $q = 6, 8, 10, 12$: the error is of order $(N/q)\tau^{2q}$ and tends to zero.
- For the same blocks with weights (1, 0.7) the error is $3.52 \cdot 10^{-3}$, $3.45 \cdot 10^{-4}$, $2.89 \cdot 10^{-5}$ at $M = 8$ and $q = 6, 8, 10$.
- For two other random normal blocks of bond dimension two on $d = 2$, with subleading eigenvalues 0.515 and 0.850 and mixed spectral radius $\tau = 0.757$, so that $|\lambda_2| = 0.850$, and weights (1, w_2) with $w_2 \in$

TNLean/MPS/Preparation/OverlappingBlockOverlap; MPSTensor.exists_norm_mpvOverlap_polarPosTensor_blockSum_sub_one_le_sq, MPSTensor.mixedTransferMatrixLeft_blockSumPosLimit_mul_mul, MPSTensor.transferMap_blockSum_embeddedBlockState and MPSTensor.im_trace_mixedMapLM_ofPhysicalMatrixLM_fixedPointTensor in TNLean/MPS/Preparation/SecondOrderBlockOverlap; IsIdempotentElem.exists_norm_trace_pow_sub_one_le_sq in TNLean/Algebra/IdempotentTracePerturbation.

⁶Formal declarations: MPSTensor.exists_approximationError_le_overlappingBlockSum_weight and MPSTensor.exists_approximationError_le_mul_overlappingBlockSum_weight in TNLean/MPS/Preparation/BlockSumError.

⁷Formal declarations: MPSTensor.nonNormalApproxOverlap_phaseBlockTensor_of_norm, MPSTensor.nonNormalApproxOverlap_phaseBlockTensor, MPSTensor.nonNormalApproxOverlap_phaseBlockTensor_norm, MPSTensor.one_sub_norm_nonNormalApproxOverlap_phaseBlockTensor_neg_one, MPSTensor.not_approximationError_le_phaseBlockTensor, MPSTensor.isBNTCanonicalForm_phaseBlockSector and MPSTensor.isBNTCanonicalForm_and_not_approximationError_le_phaseBlock in TNLean/MPS/Preparation/PhaseWeightCounterexample.

⁸Formal declarations: MPSTensor.CopyWeights.single, MPSTensor.repeatedBlockSum_single, MPSTensor.nonNormalApproxOverlap_blockSum_eq_copyApproxOverlap, MPSTensor.blockSum_comp_of_eq_zero, MPSTensor.nonNormalApproxOverlap_comp_of_eq_zero, MPSTensor.ghzAmplitude_comp_of_eq_zero, MPSTensor.exists_approximationError_le_overlappingBlockSum_complexWeight and MPSTensor.exists_approximationError_le_mul_overlappingBlockSum_complexWeight in TNLean/MPS/Preparation/BlockSumError.

$\{1, 0.99, 0.9, 0.7, 0.4, 0.1\}$, the ratio $\varepsilon/(M|\lambda_2|^q)$ stays below 0.14 for all $M \in \{1, 2, 4, 8\}$ and $q \in \{2, 4, \dots, 12\}$. For $w_2 = 0.9$ and 0.7 , $M \geq 2$, and $q \geq 4$ the error is close to $cM(w_2\tau)^{2q}$ with c between 0.4 and 1.25; for $w_2 = 0.4$ and 0.1 it no longer depends on w_2 and is that of the first block alone. Smaller weights make the overlap term smaller, and the constant of (6) does not grow as w_2 decreases.

- The overlaps z_j themselves, for three pairs of random normal blocks of bond dimension two on $d = 2$ with unit weights, with $\lambda = \max(|\lambda_2^{(1)}|, |\lambda_2^{(2)}|, \tau)$ equal to 0.490, 0.576 and 0.642: the error divided by $M\lambda^{2q}$ stays between 0.11 and 0.58 for $M \in \{1, 2, 4, 8\}$ and $q \in \{4, 6, 8, 10\}$, and for the counterexample it is 0.125 at $M = 4, 16$ and $q = 6, 8, 10$, the value $Ms^2/8$, in agreement with (7).
- For three further pairs of random normal blocks of the same kind, with $\lambda = 0.846, 0.609$ and 0.743 , and weights $(1, w_2)$, the relative error $|z_2/w_2^N - 1|/(M\lambda^{2q})$ of the overlap of the block of smaller weight stays between 0.11 and 4.7 for $w_2 \in \{1, 0.5, 0.1, 0.01\}$, $M \in \{2, 4, 8\}$ and $q \in \{4, 8\}$, except for one pair at $w_2 = 0.01$ and $q = 8$, where $w_2^N \leq 10^{-32}$ and the computation loses precision. The second-order estimate thus appears to hold relative to the weight of each block, uniformly in the weights.

10 Elimination

Two statements would repair part (ii) for multiplicity one. The first adds the hypothesis that the q -site states of distinct blocks are orthogonal, $\sum_{i_1, \dots, i_q} (A_j^{i_1} \cdots A_j^{i_q})_{ab} (A_{j'}^{i_1} \cdots A_{j'}^{i_q})_{a'b'} = 0$ for $j \neq j'$. Under it (1) holds with $|\mu_j|^q$ in place of μ_j^q : writing the blocked map as $B = \sum_j \mu_j^q B_j K_j^\dagger$ with the isometries K_j that place the bond pairs of block j , the polar decomposition of B is $V = \sum_j V_j K_j^\dagger$, $P = \sum_j \mu_j^q K_j P_j K_j^\dagger$ for $\mu_j > 0$. The states of distinct blocks, before and after the replacement by the fixed points, are then exactly orthogonal, the term $O(e^{-N/\xi_{\text{off-diag}}})$ vanishes, and part (ii) reduces to part (i) for each block, with no condition $q = o(N)$. For multiplicity one the phase of μ_j can be absorbed into A_j , which stays normal with the same transfer map, so $\mu_j > 0$ is no loss. The orthogonality at one block length q_0 also makes the mixed transfer matrices nilpotent, $E_{jj'}^{q_0} = 0$, so every λ_2 bounds their eigenvalues and this repair is a case of the second statement below. It is formalized, for every $0 < \gamma < 1$ and with a constant independent of the weights.⁹ The orthogonality holds for instance when distinct blocks use disjoint sets of physical indices.

⁹Formal declarations: `Matrix.polarIso_sum_of_orthogonal` in `TNLean/MPS/Preparation/OrthogonalSumPolar`; `MPSTensor.nonNormalApproxVector_blockSum` and `MPSTensor.mixedMapLM_eq_zero_of_conjTranspose_physicalMatrix_mul_eq_zero` in `TNLean/MPS/Preparation/OrthogonalBlockSum`; `MPSTensor.eq_zero_of_hasEigenvalue_mixedMapLM_of_conjTranspose_mul_eq_zero`, `MPSTensor.exists_approximationError_le_blockSum` and `MPSTensor.exists_approximationError_le_mul_blockSum` in `TNLean/MPS/Preparation/BlockSumError`.

The second keeps the source’s hypotheses and construction, apart from the phases of the weights, and corrects the rate: it is (6), with the correlation lengths of the mixed transfer matrices included in ξ . It is formalized for multiplicity one and all complex weights not all zero, for every $0 < \gamma < 1$ and with a constant independent of the weights, with $\beta_j = |\mu_j|^N$ in place of the coefficients μ_j^N that (S6) and (S7) take from (S4); with those coefficients the counterexample (9) refutes (S7). Blocks with multiplicity combined with overlaps are formalized for the corrected state of [con26a], which replaces (S7), for all nonzero complex weights and the coefficients $\beta_j = \sum_k \mu_{j,k}^N$ of (S4), with constants independent of the weights: the sums β_j may cancel and $\sum_j |\beta_j|^2$ may be arbitrarily small [con26c]. The proof estimates the positive part relative to the weights of the blocks, $\|(P - P_\infty)C^{-1}\| \leq Ke^{-\gamma q/\xi}$ with $C = \sum_j c_j L_j L_j^\dagger$, from a Sylvester equation for $(P - P_\infty)C^{-1}$. The state (S7) itself, with the pairs of block j on one copy k_j , is the corrected state when its coefficients are $\beta_j (c_j / \mu_{j,k_j}^q)^M$ with $c_j = (\sum_k |\mu_{j,k}|^{2q})^{1/2}$, which is $|\mu_j|^N$ for multiplicity one; with these coefficients the bounds above hold for (S7) and weights of any phase [con26a]. For multiplicity one and weight one the rate is also formalized at second order, (7), which is sharp up to $\gamma < 1$. The second-order rate for weights of different moduli and for blocks with multiplicity is not formalized: the second-order argument above uses the Lipschitz bound of the square root and the unit vectors $\iota(P)$, and for blocks of smaller weight the Gram matrices converge to a limit that is not bounded below on those blocks.

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