

One-Sided Swap Transformation of MPU Symmetries

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Verdict: local-correction (resolved).

1 Source and correction

The transformed examples in Ref. [CPGSV17, lines 2053–2063] are obtained by multiplying the periodic operator on one side by $S_N = \mathbb{S}^{\otimes N}$: $\tilde{U}^{(N)} = S_N U^{(N)}$. The proof of the swap-transformation lemma instead writes $\tilde{U}(x)^{(N)} = S_N U(x)^{(N)} S_N$ [CPGSV17, lines 2076–2084]. This conjugation is inconsistent with the stated transformed examples. The correction is to use the one-sided map $U \mapsto S_N U$ throughout that proof.

2 Operator identities

The swap is real, symmetric, unitary, and involutive. For every operator U ,

$$S_N U^\dagger S_N = U \iff (S_N U)^\dagger = S_N U, \quad (1)$$

$$S_N U^\top S_N = U \iff (S_N U)^\top = S_N U, \quad (2)$$

$$\overline{U} = U \iff \overline{S_N U} = S_N U. \quad (3)$$

The first two equivalences follow by multiplying by S_N and using $S_N^2 = I$; the third uses $\overline{S_N} = S_N$. The same involutive linear map carries a continuous path to a continuous path. Multiplication by a one-site unitary preserves unitarity of every periodic operator and canonical endpoint tensors.

The three operator equivalences and their strict path comparisons are proved by the corresponding adjunction, transposition, and conjugation results, which rest on the general transformation of paths by left multiplication with a one-site unitary and on the preservation of canonical endpoints.¹

¹Formal declarations: the results in `TNLean/MPS/MPU/Examples/SwapSymmetryTransport`; `MPOTensor.strictlyEquivalentUnderSymmetry_iff_ketLeftMul`; `MPOTensor.isMPUCanonicalForm_ketLeftMul`.

3 Scope

These identities justify the swap comparison of strict symmetry-preserving paths, with the two tensors compared after unused virtual directions are adjoined up to a common dimension, as in `docs/paper-gaps/mpu_equivalence_fixed_bond.tex`. They do not establish the stabilized comparison with general identity ancillas and blocking. The remaining questions are recorded in `docs/paper-gaps/mpu_symmetry_ancilla_transport.tex`.

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.