

The unblocked parity reading of the Fundamental Theorem of matrix product unitaries

2026-09-26

Verdict: false-source (open).

This verdict concerns the printed converse when its $U^{(N)} = V^{(N)}$ conclusion is read for the original, unblocked tensors. It does not classify the corresponding blocked-family statement as false.

1 The printed assertion

Definition SF of Cirac–Pérez-García–Schuch–Verstraete defines the standard form of the *two-site block* $\mathcal{U}_2$ of a simple tensor $\mathcal{U}$ [CPGSV17, lines 619–622]. The ensuing diagram displays $\mathcal{U}_2$ and $\mathcal{V}_2$. Nevertheless, the Fundamental Theorem of MPU states that its local unitary relations, Equations (SFuu) and (SFvv), are equivalent to $U^{(N)} = V^{(N)}$ for every N [CPGSV17, lines 624–648]. Read literally as an assertion about the original, unblocked tensors, the converse from those local relations to equality at odd lengths does not hold. The preceding diagram does, however, support a coherent reading about the blocked tensors; this note does not refute that blocked interpretation.

2 A full-rank counterexample

Take physical dimension $d = 2$ and bond dimension $D = 1$. With $i, j \in \{0, 1\}$, define two one-site tensors by

$$\mathcal{U}_{00}^{ij} = \delta_{ij}, \quad \mathcal{V}_{00}^{ij} = -\delta_{ij}. \quad (1)$$

Both have unitary periodic families. Their double-layer letters are the one-by-one scalars δ_{ij} : the two signs cancel under physical adjunction. Thus the source’s first simplicity contraction holds with $a = b = (1)$, and its second, rank-one insertion identity is automatic in the one-dimensional bond space. Both tensors are therefore simple. Their normalized transfer matrices are the one-by-one identity; they have full bond support and carry the canonical-form-II weight $\rho = (1)$.

We recall the source objects of Section III of Ref. [CPGSV17], with the leg orientation fixed in Ref. [con26]. Write $\mathcal{U}_{j,\alpha,\beta}^i$ for the entry of $\mathcal{U}^{ij}$ with left and right auxiliary indices α, β . The two *source cuts* of $\mathcal{U}$ are the flattenings

$$M_1(\mathcal{U})_{(i,\beta),(\alpha,j)} = \mathcal{U}_{j,\alpha,\beta}^i, \quad M_2(\mathcal{U})_{(\alpha,i),(j,\beta)} = \mathcal{U}_{j,\alpha,\beta}^i. \quad (2)$$

If M_1 has rank k_1 and M_2 has rank k_2 , the source chooses compact factorizations and right inverses

$$M_1 = X_1 Y_1, \quad M_2 = X_2 Y_2, \quad Y_1 Z_1 = I_{k_1}, \quad Y_2 Z_2 = I_{k_2}, \quad (3)$$

normalized by $X_1^\dagger (I_d \otimes \rho) X_1 = I_{k_1}$ and $X_2^\dagger X_2 = I_{k_2}$. The gates are the contractions

$$u_{(l,r),(i_1,i_2)} = \sum_{\beta} (Y_2)_{l,(i_1,\beta)} (Y_1)_{r,(\beta,i_2)}, \quad v_{(j_1,j_2),(r,l)} = \sum_{\alpha} (X_1)_{(j_1,\alpha),r} (X_2)_{(\alpha,j_2),l}. \quad (4)$$

Tildes denote the same objects for $\mathcal{V}$.

Neither tensor is zero, and both source cuts have rank two. In the natural physical coordinates,

$$M_1(\mathcal{U}) = M_2(\mathcal{U}) = I_2, \quad M_1(\mathcal{V}) = M_2(\mathcal{V}) = -I_2. \quad (5)$$

Choose compact rank-two factorizations

$$X_1 = X_2 = Y_1 = Y_2 = I_2 \quad \text{for } \mathcal{U}, \quad (6)$$

$$\tilde{X}_1 = \tilde{X}_2 = -I_2, \quad \tilde{Y}_1 = \tilde{Y}_2 = I_2 \quad \text{for } \mathcal{V}. \quad (7)$$

All singular values are one. The corresponding right inverses can be chosen as $Z_1 = Z_2 = \tilde{Z}_1 = \tilde{Z}_2 = I_2$; the first weighted and second ordinary X -normalizations both reduce to I_2 because $\rho = (1)$. The source gates computed from these factors have, for both tensors, the entries

$$u_{(l,r),(p,q)} = \delta_{lp} \delta_{rq}, \quad v_{(p,q),(r,l)} = \delta_{pr} \delta_{ql}. \quad (8)$$

Thus both gates are identity matrices in their displayed domain and codomain orderings; in particular both are unitary. The two minus signs on the X factors cancel in v . The printed local equations, in which x, y, z are the unitaries whose existence the Fundamental Theorem asserts, hold with $x = y = -I_2$ and $z = 1$: the product $x \otimes y$ acts trivially on u , and the two minus signs on the open half-legs cancel in the shifted gate v of Equation (SFvv). Equivalently, the two-site blocked tensors agree letter by letter,

$$\mathcal{V}_2^{(i_1,i_2),(j_1,j_2)} = (-\mathcal{U}^{i_1 j_1})(-\mathcal{U}^{i_2 j_2}) = \mathcal{U}_2^{(i_1,i_2),(j_1,j_2)}. \quad (9)$$

On a periodic chain of length N , however,

$$V^{(N)} = (-1)^N I_{2^N}, \quad U^{(N)} = I_{2^N}. \quad (10)$$

The families agree at every even length and disagree at every odd length; for example, $V^{(3)} = -I_8 \neq I_8 = U^{(3)}$.

3 Correct scope

The local equations compare standard forms of $\mathcal{U}_2$ and $\mathcal{V}_2$. Their converse can assert equality of the blocked periodic families, equivalently $U^{(2N)} = V^{(2N)}$ for positive N , but cannot by itself assert equality of the original families at every length. The forward implication from equality of the original families to the local relations is not affected. A converse about the original families needs an additional condition detecting odd lengths; it is a corrected theorem, not the printed source assertion. In particular, adding such a condition to a formal theorem must not be presented as a proof of the unqualified source-labelled equivalence. The virtual-unitary-gauge theorem obtained from equality at all lengths $N > 1$ concerns only the forward direction and remains valid.

The existing identities `MPOTensor.blockTensor_smul`, `MPOTensor.mpo_smul`, and `MPOTensor.mpo_idTensor` give a short formal route to the two-site and periodic equalities in this example. The formal parity witness records simplicity, equality of the two-site blocks, and inequality at length three. The explicit factors and local x, y, z relations above are the additional mathematical check of the printed local antecedent; they are not asserted by block equality alone.

References

- [con26] The TNLean contributors. Source cut orientation in MPU gate factorizations. Paper-gap note `mpu_source_cut_orientation`, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/mpu_source_cut_orientation.pdf.
- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.