The Hidden Range Restriction in the Source-u Contraction

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Verdict: clarification (open).

1 The graphical step

The proof of the source-tensor isometry in [CPGSV17, lines 536–557] contracts a normalized open tail between source-cut factors and then simplifies the resulting diagram. Read as an ambient matrix identity, the displayed simplification would identify the open-tail coefficient with the identity on the full product-index space. That statement is false in general: the source factors need not be onto, and only $X_2^\dagger X_2 = 1$ is available.

2 Range-restricted interpretation

The valid interpretation keeps the external source-u tensors and closes the complete network before moving the cut. The two elementary projection steps are only

$$X_1 X_1^\dagger W_\rho M_1 = M_1, \qquad\qquad X_2 X_2^\dagger M_2 = M_2, \tag{1}$$

where $W_\rho = \rho \otimes 1_d$. They follow from the source-cut factorizations and the recovery formulas for Y_1 and Y_2. Neither formula asserts an ambient coisometry.

The forward internal kernel is also orientation-sensitive. It is an output-first double-layer letter followed by the Kth power of its normalized diagonal. Conjugate transposition reverses the virtual chain, so the reflected simple-contraction equations cannot be inserted into this coefficient pointwise. The remaining independent obligation is the equality between the ordinary source-u Gram matrix and the normalized closed $(K+2)$-site output tail, proved with the full external network retained.

3 Verdict

Classification: local fix (graphical clarification). The source argument is read as a range-restricted closed-network contraction, not as an identity on the ambient product-index space. This does not classify the source lemma as false: its diagram retains the external network, but that restriction must not be discarded when translating the argument into indexed formulas.

The forward kernel and its output-layer expression are formalized by `MPOTensor.sourceUForwardKernel` and `MPOTensor.sourceUForwardKernel_eq_outputLayer_mul_normalizedDiagonal_pow`. The range identities are `MPOTensor.sourceX_mul_conjTranspose_mul_weight_mul_sourceCutM` and `MPOTensor.sourceX_mul_conjTranspose_mul_sourceCutM`. The fully expanded normalized metric is `MPOTensor.IsMPU.normalized_sourceU_openTail_metric`. This last theorem does not identify that metric with the ordinary source-u Gram matrix; the final closed-network equality remains a separate obligation.

References

[CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.