

Supplied Two-Site Decompositions of the Shift Matrix Product Unitaries

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Verdict: scope-restriction (open).

1 Source statement

Definition SF of Ref. [CPGSV17] defines the standard form of the two-site block $\mathcal{U}_2$ of a simple tensor $\mathcal{U}$ in canonical form II. Its gates u and v are the unitaries obtained from the source cuts of $\mathcal{U}$.

Let $\mathbb{S}$ be the swap of two d -dimensional spins, with entries $\mathbb{S}_{(p,q),(r,s)} = \delta_{p,s}\delta_{q,r}$ on $\mathbb{C}^d \otimes \mathbb{C}^d$. For the three shift families of Section VII, the source says that the gates “can be easily derived” from the figure of the shifted spins and displays

$$u_1 = \mathbb{1} \otimes \mathbb{1}, \quad v_1 = \mathbb{1} \otimes \mathbb{1}, \quad (1)$$

$$u_3 = \mathbb{S}, \quad v_3 = \mathbb{S}, \quad (2)$$

together with the blocked gates $u_2^{(2)}, v_2^{(2)}, u_3^{(2)}, v_3^{(2)}$ of the two counterpropagating families. For the swap-transformed families it displays $\tilde{u}_1 = \mathbb{S}$, $\tilde{v}_1 = \mathbb{1} \otimes \mathbb{1}$, $\tilde{u}_2 = \mathbb{1} \otimes \mathbb{1}$, and $\tilde{v}_2 = \mathbb{S}$.

2 Formalized scope

For each displayed pair of gates, the formalization constructs an explicit two-site decomposition with supplied gates. The gates are the displayed unitaries, the half factors are normalized, and their open contraction recovers the tensor.

For the blocked counterpropagating families, the half factors are pulled back from an explicit source-cut factorization with trace-one weight $d^{-2}\mathbb{1}_{d^2}$. The gates are the source gates of that factorization.

For U_1 , the unblocked U_3 , $\tilde{U}_1$, and $\tilde{U}_2$, the one-site tensor of physical dimension d^2 is read as a pair of spins and decomposed directly. These decompositions are not identified with the standard form of Definition SF: the tensor is not presented as the two-site block of a simple canonical-form-II tensor, and its gates are not identified with the source gates of a source-cut factorization.

The constructions are `MPOTensor.shiftExampleUStandardForm`, `MPOTensor.shiftExampleUStandardForm`, `MPOTensor.shiftExampleUBlockedStandardForm`, `MPOTensor.shiftExampleUBlockedStandardForm`, `MPOTensor.shiftExampleTildeUStandardForm`, and `MPOTensor.shiftExampleTildeUStandardForm` in `TNLean/MPS/MPU/Examples/ShiftStandardForms.lean`.

3 Elimination plan

For each unblocked family, present the one-site tensor as the two-site block of a simple tensor in canonical form II, or block it further as done for the counterpropagating families. Then identify the supplied gates with the source gates of an explicit source-cut factorization of that tensor.

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.