The Supplied Fixed Pair in the Reduced-Representative Step of Proposition IV.5

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Verdict: scope-restriction (open).

1 Source statement and derivation of the fixed pair

Proposition IV.5 of Cirac–Pérez-García–Schuch–Verstraete states that for a continuous map of tensors $[0,1] \ni x \mapsto \mathcal{W}(x)$, each generating MPUs but not necessarily in canonical form, the ranks $r(x)$ and $\ell(x)$ of the associated simple tensors are constant [CPGSV17, Proposition IV.5]. The proof begins with a single tensor $\mathcal{W}$ generating an MPU and derives, from the MPU property alone and after blocking D^4 sites, that the blocked canonical form is simple and that its transfer operator is rank one: as a completely positive map it is

$$X \longmapsto \operatorname{tr}(LX)\, R, \tag{1}$$

with positive semidefinite left and right fixed points $L \succeq 0$ and $R \succeq 0$ normalized by $\operatorname{tr}(LR) = 1$. The proof then reads off, from the displayed factorization through an isometry V, that the tensor absorbs the support projections on both sides,

$$\mathcal{W}P = \mathcal{W} = Q\mathcal{W}, \tag{2}$$

where P and Q are the orthogonal projections onto the supports of L and R, respectively. The source treats all four ingredients—the rank-one transfer formula, the positive semidefiniteness, the normalization, and the two-sided support absorption—as derived consequences of blocking an arbitrary MPU tensor.

2 The formal scope issue

The formal reduced-representative calculation reverses the direction of this part of the argument: it takes the fixed pair and its consequences as explicit hypotheses rather than deriving them. The letterwise absorptions $W^{ij}P = W^{ij}$

and $QW^{ij} = W^{ij}$, the rank-one transfer formula $E_W(X) = \mathrm{tr}(LX)R$, the positivity $L, R \succeq 0$, and the normalization $\mathrm{tr}(LR) = 1$ are all assumptions of the formal statements. No hypothesis is inferred from the bare MPU predicate; the module docstring records this candor in the sentence that all fixed-point and support hypotheses are explicit.

What the formal results prove is the remainder of the source argument, starting from the supplied pair. With $T = \widetilde{P}$ the reduced projection of P and Q, the reduced tensor $\widetilde{W^{ij}} = TW^{ij}T$ generates the same periodic MPO at every system size, its transfer operator has the compressed rank-one formula with left and right fixed points TLT and TRT, the compression preserves the trace normalization $\mathrm{tr}((TLT)(TRT)) = 1$, and in isometric coordinates $V^\dagger V = I$ with $VV^\dagger = T$, both compressed fixed matrices $V^\dagger LV$ and $V^\dagger RV$ are positive definite.

The restriction is not vacuous. The source derivation of the fixed pair uses the blocking to D^4 sites and the simple canonical-form structure of the blocked tensor; the unblocked tensor, or an arbitrary representative of it, carries no such pair without that construction. Until the supplier is formalized, the formal statements apply only to tensors already equipped with the fixed pair, whereas the source claims the conclusions for every MPU tensor after blocking.

3 Affected declarations

The declaration `MPOTensor.mpo_virtualSandwich_reducedProjection_eq` assumes the orthogonal-projection property of P and the letterwise absorptions $W^{ij}P = W^{ij}$ and $QW^{ij} = W^{ij}$. `MPOTensor.transferMap_virtualSandwich_reducedProjection` assumes the rank-one transfer formula. The declarations `MPOTensor.trace_compressed_fixed_pair_reducedProjection` and `MPOTensor.compressed_fixed_pair_posDef_reducedProjection` assume, respectively, that P and Q are orthogonal projections, the two-sided support absorptions of L and R, and $\mathrm{tr}(LR) = 1$; and that $L, R \succeq 0$ have support projections P, Q. The latter also assumes an isometry V onto the range of T.

The comparison declaration `MPOTensor.virtualSandwich_hat_eq_reducedProjection` assumes the orthogonal-projection property of P, the factor equations $XL = P$, $RZ = Q$, and $PQY = T$, and the letterwise absorptions. The rank equalities `MPOTensor.rightRank_hat_eq_reducedProjection` and `MPOTensor.leftRank_hat_eq_reducedProjection` use the same supplied data and additionally assume that X, Z, and Y are invertible. Finally, `MPOTensor.exists_units_for_hat_reducedProjection` assumes only $L, R \succeq 0$ and supplies the invertible ambient support-inverse extensions X and Z, together with an invertible right factor Y satisfying $PQY = T$. Its matrix theorem, `Matrix.exists_units_supportInvExtension_reducedProjection_rightFactor`, has the same positive-semidefinite hypotheses and assumes no fixed-point, normalization, or absorption identity.

4 Elimination plan

The restriction is removed by formalizing the missing supplier: the blocking to D^4 sites of an arbitrary MPU tensor, the two-projection Jordan decomposition of the resulting support projections, and the isometric factorization that yields the letterwise absorptions and the rank-one transfer formula with positive semidefinite normalized fixed points. This is tracked by issues #6137 and #6022: the former constructs and transports the reduced representative and supplies the positive fixed pair with the letterwise absorptions for the common blocked tensor, and the latter carries the full continuity-index route of which Proposition IV.5 is a part. Until these are closed, no formal statement may infer the fixed pair or the absorptions from the bare MPU predicate.

5 Verdict

Verdict: scope restriction. The paper derives the fixed pair $L, R \succeq 0$ with $\mathrm{tr}(LR) = 1$, the rank-one transfer formula, and the two-sided support absorption $\mathcal{W}P = \mathcal{W} = Q\mathcal{W}$ from the MPU property after blocking D^4 sites. The present formalization states the reduced-representative conclusions under these ingredients as explicit hypotheses and does not construct the supplier. The results are correct as stated, but narrower than the source argument; the hypotheses become theorems only after issues #6137 and #6022 are resolved.

References

[CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.