MPU One-Letter Mixed Residual Trace: Resolved Local Correction

2026-08-09

Verdict: local-correction (resolved).

1 Decomposition and source assertion

Let $\mathcal{U}$ be a matrix product unitary with physical dimension $d > 0$, and let $\mathcal{W}$ be its physical-adjoint double layer. The source writes the local tensor in the identity-plus-traceless form

$$\mathcal{W} = E \otimes \mathrm{Id} + \sum_{\alpha} S_\alpha \otimes \sigma_\alpha, \tag{1}$$

where E and the residual coefficients S_α act on the doubled virtual space and every physical matrix σ_α is traceless. In the basis-free formulation, contraction against a physical matrix X is

$$\mathrm{contr}_{\mathrm{phys}}(\mathcal{W}, X) = \mathrm{tr}(X)E + R_{\mathcal{W}}(X), \tag{2}$$

$$R_{\mathcal{W}}(X) = \mathrm{contr}_{\mathrm{phys}}\left(\mathcal{W}, X - \frac{\mathrm{tr}(X)}{d}\,\mathrm{Id}\right). \tag{3}$$

Thus $R_{\mathcal{W}}(X)$ is a residual slice, and the matrices S_α are obtained by choosing traceless basis arguments.

The proof of Proposition III.3(ii) states that every positive-length closed product containing at least one residual coefficient has vanishing trace [CPGSV17, lines 405–409]. In particular,

$$\mathrm{tr}(S_\alpha) = 0, \tag{4}$$

and more generally every mixed word in E and the S_α with at least one residual letter has zero trace.

2 Restricted auxiliary theorems and shifted-nilpotence recovery

The definition of MPU unitarity directly supplies closed-chain identities at lengths $N > 1$. Accordingly, the intermediate constant-residual and mixed-residual auxiliary theorems first prove

$$\operatorname{tr}(R_{\mathcal{W}}(X_0)\cdots R_{\mathcal{W}}(X_{N-1})) = 0 \qquad (N > 1), \tag{5}$$

and the analogous identity for mixed identity and residual factors. [1]

The apparent one-letter gap is recovered from these longer words. Fix a physical matrix X and set $A = R_{\mathcal{W}}(X)$. Repeating X at every site in the constant-residual auxiliary theorem gives

$$\operatorname{tr}(A^N) = 0 \qquad (N > 1). \tag{6}$$

The shifted zero-trace-power theorem implies that A is nilpotent. Since the trace of a nilpotent complex matrix vanishes,

$$\operatorname{tr}(A) = 0. \tag{7}$$

The formal one-letter theorem carries out exactly this argument. [2] The full positive-length theorem then separates $N > 1$ from $N = 1$; in the one-letter branch the unique physical site is residual and the preceding identity applies. [3]

This reverses the local order of the source proof: the formal argument proves the longer closed words first and derives the one-letter trace afterward. It does not strengthen the MPU hypotheses.

3 Verdict

Verdict: local correction (resolved). The intermediate auxiliary theorems retain the explicit scope restriction $N > 1$, but shifted nilpotence derives the missing $N = 1$ case. The final positive-length mixed-residual theorem has the same hypotheses and conclusion as the cited source assertion. The former obligation is recorded in TNLean issue #5807, which was closed after this derivation was formalized.

References

[CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.

[1] Formal declarations: `MPOTensor.IsMPU.trace_prod_residualSlice_doubleLayerTensor_eq_zero` and `MPOTensor.IsMPU.trace_prod_mixedResidualFactor_doubleLayerTensor_eq_zero`.

[2] Formal declarations: `MPOTensor.IsMPU.trace_residualSlice_doubleLayerTensor_eq_zero` and `Matrix.isNilpotent_of_forall_trace_pow_eq_zero_of_one_lt`.

[3] Formal declaration: `MPOTensor.IsMPU.trace_prod_mixedResidualFactor_doubleLayerTensor_eq_zero_of_pos`.