

Nil-Matrix Product-Length Bound in the MPU Blocking Argument

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Verdict: local-correction (resolved).

1 The source assertion

The blocking argument for matrix product unitaries in [CPGSV17, lines 405–426] considers a possibly nonunital associative subalgebra A of the endomorphisms of a complex vector space of dimension D^2 . Every element of A is nilpotent. The source invokes results of Nagata–Higman and Razmyslov to choose a mixed-product length J' satisfying

$$J' < D^2, \quad A^{J'} = 0. \quad (1)$$

Here $A^{J'} = 0$ means that every ordered product of J' elements of A vanishes.

The cited nil-algebra references are Nagata, *Journal of the Mathematical Society of Japan* 4 (1952), 296–301; Higman, *Proceedings of the Cambridge Philosophical Society* 52 (1956), 1–4; and Razmyslov, *Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya* 38 (1974). These results provide nilpotency theorems and general quantitative bounds, but do not supply the strict linear estimate in (1).

2 The corrected nil-matrix bound

Let V be an n -dimensional vector space over a field and let $A \subseteq \text{End}(V)$ be a possibly nonunital associative subalgebra consisting entirely of nilpotent endomorphisms. The classical finite-dimensional nil-matrix theorem, also known as Levitzki’s theorem and equivalently obtainable from Engel’s theorem by simultaneous strict triangularization, gives

$$A^n = 0. \quad (2)$$

Thus the concrete transfer-space algebra in the source satisfies

$$J' \leq D^2. \quad (3)$$

The weak inequality is sharp in general: the algebra of strictly upper-triangular $n \times n$ matrices admits a nonzero ordered product of $n - 1$ matrix units. Accordingly, no general replacement of (3) by $J' < D^2$ is available.

3 Effect on the blocking estimate

The source has already obtained a first Jordan-block length J satisfying $J < D^2$. Combining this strict inequality with the corrected weak bound gives

$$JJ' < D^2 D^2 = D^4. \quad (4)$$

Therefore the final strict blocking estimate is unchanged. Only the intermediate claim $J' < D^2$ is replaced by $J' \leq D^2$.

The formal development proves (2) for arbitrary finite-dimensional vector spaces and then transports it to square matrices.¹ It includes the zero-dimensional case and fixes the ordered-product convention. The strict-upper-triangular example is recorded here only as the mathematical reason that a strict bound is unavailable; no formalization of that example is claimed.

4 Classification

Verdict: local correction. Replacing the unsupported intermediate strict bound by the finite-dimensional nil-matrix bound preserves every hypothesis and conclusion needed by the surrounding blocking argument, and (4) retains the source’s final strict estimate.

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.

¹Formal declarations: `NonUnitalSubalgebra.list_prod_eq_zero_of_forall_isNilpotent` and `NonUnitalSubalgebra.matrix_list_prod_eq_zero_of_forall_isNilpotent` in `TNLean/Algebra/NilMatrixSubalgebra`.