

CZX: Normalization and the Physical Z Gates

2026-09-04; updated 2026-09-27

Verdict: false-source (resolved).

1 The 2017 contraction convention and its obstruction

Section II, “Basics in Matrix Product Vectors”, of [CPGSV17] defines the periodic contraction as an ordinary trace, with no length-dependent prefactor. Section III, “Structure of MPU”, requires the resulting operator to be unitary and separately normalizes its transfer matrix by $1/q$, where q is the local physical dimension ($q = 2$ here). That transfer normalization does not rescale the physical operator.¹

Write $u, d, \ell, r \in \{0, 1\}$ for upper, lower, left and right indices, $X_{ud} = \delta_{u,1-d}$, and $H_{\ell r} = 2^{-1/2}(-1)^{\ell r}$. The example headed “CZX” in the time-reversal symmetry subsection of [CPGSV17] prints²

$$P_{\ell r}^{ud} = X_{ud} \delta_{u,\ell} H_{\ell r}. \quad (1)$$

For binary output and input configurations a, b of length $n > 0$, with cyclic indices $a_n = a_0$, the copy deltas force the unique virtual path $\ell_j = a_j$. Consequently

$$\text{MPO}_n(P)_{a,b} = 2^{-n/2} \prod_{j=0}^{n-1} \delta_{a_j, 1-b_j} (-1)^{\sum_{j=0}^{n-1} a_j a_{j+1}}. \quad (2)$$

Every row and column has one nonzero entry, of modulus $2^{-n/2}$, so $\text{MPO}_n(P) \text{MPO}_n(P)^\dagger = 2^{-n} I$, not I . At $n = 2$ the row indexed by 00 has only the entry $\text{MPO}_2(P)_{00,11} = 1/2$; its squared norm is $1/4$. Blocking $L > 0$ sites replaces n by Ln and cannot remove this obstruction. This is a nondegenerate normalization error, not a zero-sector convention.

¹Source traceability: `Papers/1703.09188/paper_v2.tex`, lines 226–246 and 318–350. The declared auxiliary-label build for this archived source currently fails at the title’s malformed accent command; the printed section and example designations are used here instead.

²Source traceability: `Papers/1703.09188/paper_v2.tex`, lines 1914–1930.

2 What the 2025 diagram actually says

Equations (131) and (132) of [FRBC25] define the blue two-leg tensor as $W = \sqrt{2}H$, with entries $W_{\ell r} = (-1)^{\ell r}$; the wire junctions are ordinary copy deltas. The CZX tensor in Equation (160) of [FRBC25] has two such blue tensors, two lower physical X gates and two upper physical Z gates. None may be omitted.³

For comparison only, let $A = \sqrt{2}P$. Reading the two physical legs left to right, let their outputs be a, b and inputs c, d . Writing P_2 and A_2 for the ordinary two-site blocks of P and A , the latter has entries

$$(A_2)_{\ell r}^{ab, cd} = \delta_{a, 1-c} \delta_{b, 1-d} \delta_{\ell, a} (-1)^{ab+br}. \quad (3)$$

Since $Z_{aa} = (-1)^a$, the displayed 2025 tensor is instead

$$B_{\ell r}^{ab, cd} = \delta_{a, 1-c} \delta_{b, 1-d} \delta_{\ell, a} (-1)^{a+b+ab+br} = (-1)^{a+b} (A_2)_{\ell r}^{ab, cd}. \quad (4)$$

Thus $B = (Z \otimes Z)A_2$ on the output physical leg, equivalently the block of the one-site tensor with entries $(-1)^u A_{\ell r}^{ud}$. It is neither P_2 nor A_2 in these coordinates. For example, at $(a, b, c, d, \ell, r) = (0, 1, 1, 0, 0, 0)$, their entries are respectively $B = -1$, $P_2 = 1/2$, $A_2 = 1$. On a ring of m blocked sites the additional factor is $(-1)^{\sum_{j=0}^{2m-1} a_j}$, which is not a constant phase. At $m = 2$, compare all-zero output with output 0100, each against its bitwise-complement input. This also excludes an identification merely by periodic virtual gauge transformation or a global scalar.

The left-to-right pair (a, b) is encoded as $2a+b$; successive pairs give the four-bit order used for two neighboring blocked sites.⁴ The comparison (4) is now checked in three steps: the shared tensor is the ordinary two-site block of the decorated one-site tensor, that one-site tensor is $(-1)^u$ times the unnormalized review tensor A , and the printed tensor is $2^{-1/2}A$.⁵

3 Checked boundary and elimination plan

The checked witness is exactly the printed tensor (1). The formal results establish its two-site all-zero row formula and its failure to generate a matrix product unitary.⁶ The all-length contraction (2) is a mathematical derivation in this note, not an additional checked theorem.

The literal printed cross-paper tensor identity is false. The corrected target is the exact Z -decorated tensor (4), with its all-length monomial formula and unitarity, using ordinary blocking and the stated bit ordering. This target is formalized: its coordinates, its all-length monomial formula, and the unitarity

³Source traceability: `Papers/2502.20257/main.tex`, lines 3795–3851 and 4503–4547.

⁴The existing encodings are `MPOTensor.CZX.siteBits` and `MPOTensor.CZX.localBits`.

⁵Results: `MPOTensor.CZX.tensor`, `MPOTensor.blockTwo_eq_blockTensor_reindex`, `CZXCompression.decoratedSiteTensor_eq_smul_reviewCZXTensor` and `CZXCompression.printed2017Tensor_eq_smul_reviewCZXTensor`.

⁶Witness: `MPOTensor.CZX.printed2017Tensor`. Results: `MPOTensor.CZX.printed2017Tensor_two_zero_row` and `MPOTensor.CZX.printed2017Tensor_not_isMPU`.

of its periodic operator at every positive length are checked, which resolves the note.⁷ Any connection to the 2017 gate pair $u = CZ(X \otimes X), v = CZ$, where $CZ = \text{diag}(1, 1, 1, -1)$, of [CPGSV17] must explicitly specify the normalization and physical change of representative, rather than assert literal equality. No classification signs, action data, defect tensors, or completion certificates are supplied here. This audit does not establish the two original CZX classification claims.

4 The 2020 review repeats the printed tensor

Appendix A, “The CZX MPU”, of [CPGSV21] writes the CZX matrix product unitary as $|0\rangle\langle 1| \otimes |0\rangle\langle +| + |1\rangle\langle 0| \otimes |1\rangle\langle -|$, with the normalized vectors $(\pm| = 2^{-1/2}(|0| \pm |1|)$ on the virtual level. Its two nonzero matrices are $2^{-1/2}A^{01}$ and $2^{-1/2}A^{10}$, where

$$A^{01} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^{10} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad (5)$$

are the matrices the review prints two displays later. The bra-ket display is exactly the tensor (1), and the printed matrices (5) are its rescaling $A = \sqrt{2}P$.⁸

Every later claim of the passage holds for the matrices (5): the operator $O(A)$ is unitary and normal after blocking two sites, its square has the displayed bond-four matrices with integer entries, the invariant subspace of the projector P and the second reduction step with the projector Q lead to the canonical form $B^{ij} = (-1)\delta_{ij}$, and $O(A)^2 = (-1)^N I$.⁹ For the normalized tensor of the bra-ket display the periodic operator is $2^{-N/2}O(A)$, so its square is $2^{-N}(-1)^N I$ and the global identity fails by the factor 2^{-N} , as in (2). The formalization takes the printed matrices (5) as the source tensor of the review and records the bra-ket display as this normalization slip.

A smaller discrepancy of the same passage is recorded here.

Gate order. The review describes the operator in words as controlled- Z gates between all adjacent sites “followed by a Pauli X on all sites”, which in time order is $X^{\otimes N} D_N$, where D_N is the diagonal operator with entries $(-1)^{\sum_n s_n s_{n+1}}$ on a ring of N sites; this is also the order $U_{CZX} = U_X U_{CZ}$ of [CLW11]. The printed matrices (5) contract instead to $O(A) = D_N X^{\otimes N}$. Since $D_N X^{\otimes N} =$

⁷Results: `MPOTensor.CZX.tensor_apply`, `MPOTensor.CZX.mpo_tensor_apply`, `MPOTensor.CZX.mpo_tensor`, `MPOTensor.CZX.mpo_tensor_mem_unitaryGroup` and `MPOTensor.CZX.tensor_isMPU`; the work was tracked by GitHub issue #7738 and GitHub issue #7752, both closed. The one-site representative of [GRS24], which is the same one-site tensor with its two bonds exchanged, is formalized with its operator, fusion and normality results through `CZXCompression.czxDecoratedTensor_eq_transpose`.

⁸Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2582–2601. Witnesses: `CZXCompression.reviewCZXTensor` and `CZXCompression.printed2017Tensor_eq_smul_reviewCZXTensor`.

⁹Results: `CZXCompression.reviewCZXTensor_isMPUPos`, `CZXCompression.reviewCZXMPs_isNormal`, `CZXCompression.reviewCZXSquare_apply_zero`, `CZXCompression.reviewQ_conj_reviewBlock` and `CZXCompression.mpo_reviewCZXTensor_mul_self`.

$(-1)^N X^{\otimes N} D_N$, the two readings differ by the global sign $(-1)^N$. Both square to $(-1)^N I$, so none of the later claims of the passage depends on the choice; the formalization follows the printed matrices.¹⁰

5 The review’s two statements on the square

The main text of [CPGSV21], in the discussion of two-dimensional symmetry-protected phases, writes the CZX symmetry as $O_Z = \bigotimes_i CZ_{i,i+1} \otimes \bigotimes_i X_i$ and states that the square of O_Z “has a 1-dimensional invariant subspace equal to $O_1 = I$ ”.¹¹ Appendix A carries the reduction out: after the projectors P and Q the one-dimensional block is the bond-one tensor $B^{ij} = (-1)\delta_{ij}$, which “globally means $O(A)^2 = (-1)^N I$ ”.¹²

The two statements differ by the sign $(-1)^N$. The bond-one tensor $(-1)\delta_{ij}$ generates $(-1)^N I$, which is the identity operator O_1 exactly when N is even. Both gate orders of O_Z square to $(-1)^N I$, by the commutation relation of the previous section, so the discrepancy does not come from the ambiguous product in the main text. It is a normalization of the tensor: multiplying the CZX tensor by a scalar c multiplies its square by c^{2N} , so with $c^2 = -1$ (for instance $c = i$) the reduced block is δ_{ij} and the square is exactly $O_1 = I$ at every length. The main-text sentence identifies the block with O_1 up to this length-dependent phase, the freedom usually allowed for matrix product operator representations of a group; read literally at odd N , with the tensor of Appendix A, it is false. This note classifies the discrepancy as a clarification, not a source error. The formalization follows the appendix: for the printed matrices the square is $(-1)^N I$ at every positive length.¹³

References

- [CLW11] Xie Chen, Zheng-Xin Liu, and Xiao-Gang Wen. 2D symmetry protected topological orders and their protected gapless edge excitations. *Phys. Rev. B*, 84:235141, 2011.
- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair

¹⁰Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2586–2591; `References/1106.4752/source/dDSPTmodel.tex`, line 287. Results: `CZXCompression.mpo_reviewCZXTensor_apply` and `CZXCompression.mpo_reviewCZXTensor_eq_smul`.

¹¹Source traceability: `Papers/2011.12127/TN-Review-main.tex`, line 1472.

¹²Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2596–2601.

¹³Results: `CZXCompression.reviewQ_conj_reviewBlock`, `CZXCompression.mpo_mulTensor_reviewCZXTensor` and `CZXCompression.mpo_reviewCZXTensor_mul_self`.

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[FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.

[GRS24] José Garre-Rubio and Norbert Schuch. Fractional domain wall statistics in spin chains with anomalous symmetries. arXiv:2405.00439v2, 2024.