Full Support in MPU Normality and Theorem III.8

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Verdict: scope-restriction (open).

1 Source statement and surrounding convention

The normal-tensor proposition of Cirac–Pérez-García–Schuch–Verstraete states that the normalized MPU transfer matrix has one nonzero eigenvalue, equal to one, and that the normalized local tensor is normal [CPGSV17, Proposition III.1]. The proposition itself displays no canonical-form or minimality hypothesis. Its section has already imposed the standing convention that, without further notice, the local tensor has first been replaced by a tensor in canonical form [CPGSV17, discussion preceding Proposition III.1]. The preceding canonical-form definition makes this tensor a direct sum of irreducible positive-dimensional blocks and permits replacement by another representative generating the same matrix product vectors.

2 The formal support issue

The literal CPSV canonical-form data used in the formalization permits a coisometric embedding of the retained direct sum into a larger ambient bond space. It also permits displayed blocks with zero weight. These choices are useful for exact reconstruction. The paper's literal canonical-form definition also permits zero-weight displayed blocks, so canonical-form membership alone does not discard these coordinates. The proof of the proposition is correct for the intended reduced representative in which zero-weight blocks and any ambient zero complement have been omitted, but the source does not state this reduction as a separate step and the formalization does not yet construct it.

This distinction is necessary. Starting with any MPU representative, adjoin a zero virtual direct summand. Every periodic contraction is unchanged, hence the enlarged tensor still generates the same unitary operators for every length. Nevertheless, the projection onto the original virtual summand is a nontrivial invariant projection, so the enlarged ambient tensor is not irreducible and therefore is not normal. Thus the bare assertion "MPU implies normal tensor" is false for nonminimal ambient representatives.

The formal support condition is

$$\sum_{k:\, \mu_k \neq 0} D_k = D, \tag{1}$$

where D is the ambient bond dimension. If there is exactly one active block, this equality says that the block itself has dimension D. Positivity of all retained block dimensions and the total-dimension bound then show that no other retained block exists. The ambient coisometry is square and hence unitary; block assembly and unitary conjugation transport irreducibility from the normal block to the ambient tensor.

3 The source factors and Theorem III.8

Let $\mathcal{U}$ be an MPU tensor with physical dimension d and bond dimension D, and let E be the transfer matrix of the normalized tensor $d^{-1/2}\mathcal{U}$. After the normal-tensor proposition, the source assumes without loss of generality that this normalized tensor is in canonical form II [CPGSV17, discussion following Proposition III.1]. Choose such data with full active support. Let $\mathcal{A}_1$ be its unique active block, let V be the inclusion of that block into the ambient bond space, and let Λ be the block's diagonal positive right fixed point normalized to have trace one. Set

$$\rho = V\Lambda V^\dagger. \tag{2}$$

The source factors are defined using this matrix ρ. In the first singular value decomposition, let V_1 denote the coisometry appearing in $M_1 = V_1^\dagger D_1 U_1$. The source orders the product index as (j, α) and defines the compressed positive metric

$$G = V_1(\mathbb{1}_d \otimes \rho)V_1^\dagger. \tag{3}$$

The normalization factor appearing in X_1 is $G^{-1/2}$, not G. The formal source matrix orders the same index as (α, j). Let

$$\chi\colon \mathbb{C}^D \otimes \mathbb{C}^d \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^D, \qquad \chi(e_\alpha \otimes e_j) = e_j \otimes e_\alpha. \tag{4}$$

In the virtual–physical order, the source weight is therefore

$$W_\rho = \chi^\dagger(\mathbb{1}_d \otimes \rho)\chi = \rho \otimes \mathbb{1}_d, \tag{5}$$

and the compressed positive metric is

$$A = V_1 W_\rho V_1^\dagger. \tag{6}$$

The corresponding normalization factor is $A^{-1/2}$. The two Kronecker orders are related by the product-index equivalence and are not literally the same matrix in one fixed ordering.

The inverse square root only requires the compressed metric $V_1 W_\rho V_1^\dagger$ to be positive definite; it does not by itself force ρ to be positive definite on the ambient bond space. The current TNLean source-factor construction assumes ambient positive definiteness, and full active support supplies that stronger hypothesis [CPGSV17, Section III, source-factor construction]. The same factors define u and v in Lemma III.7 and Theorem III.8. Thus ambient positive definiteness is a restriction of the current formal construction, not a logical consequence of invertibility of the compressed source metric or of the bare MPU predicate.

Let $(\Phi|$ denote the vectorized identity bra $\mathrm{vec}(\mathbb{1}_D)^\mathsf{T}$. The supported formal route has two bond-dimension branches. If $D = 1$, the theorem `MPOTensor.IsMPU.normalized_transfer_matrix_eq_one_fin_one` supplies

$$J = 1, \qquad \rho = \mathbb{1}_1, \qquad E^J = |\rho)(\Phi|. \tag{7}$$

If $D > 1$, the route assumes

$$\mathcal{C} \in \mathrm{CFII}(d^{-1/2}\mathcal{U}), \tag{8}$$

$$\sum_{k:\,\mu_k \neq 0} D_k = D. \tag{9}$$

Here $\mathcal{C}$ denotes the chosen canonical-form-II data, and the second identity is full active support. The theorem `MPOTensor.IsMPU.normalized_transfer_power_eq_vecMulVec_of_reduced_cfii` then supplies the unique active block $\mathcal{A}_1$, the matrices Λ and ρ, and the identities

$$\rho = V\Lambda V^\dagger, \tag{10}$$

$$\rho > 0, \tag{11}$$

$$\mathrm{tr}(\rho) = 1, \tag{12}$$

$$E|\rho) = |\rho), \tag{13}$$

$$(\Phi|E = (\Phi|, \tag{14}$$

$$E^J = |\rho)(\Phi|, \qquad J = D^2 - 1. \tag{15}$$

In both branches, the positive trace-one matrix ρ is the source weight used to define the factors and the operators u and v.

The explicit reduced-CFII and full-active-support hypotheses resolve the ambient-positivity question tracked by issue #5855. They do not prove the two complete network identities needed for Theorem III.8. The source-u range-insertion equality remains tracked by issue #5982 [CPGSV17, Lemma III.7]. The complete source-v equivalence remains tracked by issue #6071 [CPGSV17, proof of Theorem III.8].

For $D > 1$, reduced canonical-form-II data with full active support are assumed for the same tensor $\mathcal{U}$ whose source cuts, ranks, and source operators occur in the two planned results. For $D = 1$, the one-dimensional stabilization theorem supplies the source pair directly. In either case the datum provides $J > 0$, a positive trace-one matrix ρ, and

$$E^J = |\rho)(\Phi|. \tag{16}$$

The admissible source-isometry and simple-tensor equivalence nodes cite Lemma III.7 and Theorem III.8 directly. Both retain the source factors of $\mathcal{U}$.

Blocking appears only inside the proof of the admissible source-isometry lemma. Let J be the supplied stabilization exponent, put $K = JD^2$, and let $\mathcal{U}^{[K]}$ denote the direct K-site block of $\mathcal{U}$. This internal block supplies the direct and reflected contractions used to evaluate the closed network for $u^\dagger u$. It is not used to replace the source cuts, ranks, or operators of $\mathcal{U}$, and no transport of source data across blocking is asserted.

4 Downstream standard-form, index, and symmetry scope

The Chapter 28 blueprint records the preceding hypotheses as a reduced full-support source datum for the same tensor. This is an explicit input, not an existence theorem. It retains unrestricted, unformalized source entries under the original labels `lemuisometry`, `ThmFund1`, `SF`, `def:index`, `index-well-defined`, `lem:mpu_index_tensor`, and `lem:mpu_index_composition`. The current reduced full-support statements have the distinct labels `lem:mpu_admissible_source_u_isometry`, `thm:mpu_admissible_simple_tensor_equivalence`, `def:mpu_admissible_standard_form`, `def:mpu_admissible_index`, `prop:mpu_admissible_index_well_defined`, `lem:mpu_admissible_index_tenso` and `lem:mpu_admissible_index_composition`.

Chapter 28 contains thirty-nine labelled `mpu_admissible` entries. Thirty-eight are restricted forms of results or definitions in [CPGSV17]; each carries its own direct citation or exact source-line locator. The remaining entry, `def:mpu_admissible_equivalence_datum`, is an auxiliary hypothesis introduced here. It cites the source path arguments that motivate its conditions without attributing the datum itself to the paper.

The unrestricted source Fundamental Theorem of MPU remains under `FundamentalMPU` and depends only on the source standard form `SF`. The separate Admissible Fundamental Theorem depends on the admissible standard form and the admissible simple-tensor equivalence theorem. Likewise, the unrestricted source Index Theorem remains under `IndexTh` and depends on the source index, source well-definedness, unrestricted tensor-product and composition additivity, source continuity, and source equivalence. The common-block reduced-data versions of the two additivity statements have distinct admissible labels and feed only the Admissible Index Theorem. That theorem records the result currently justified by the formal interfaces. It has a source-data restriction at every blocking length where Theorem III.8 is used. A blocking length is admissible only when the actual blocked tensor is simple and carries its own source datum. Well-definedness compares two admissible simple blockings. Tensor-product additivity requires one common blocking length at which both factors and their tensor product are simple and carry source data. The published composition proof first uses $\mathcal{U}_k$, $\mathcal{U}'_k$, and $\mathcal{W}_k$, then writes $\mathcal{U}_{2k}$ and $\mathcal{U}'_{2k}$ in standard form and computes the ranks of $\mathcal{W}_{4k}$. The admissible theorem therefore requires separate source data for all six of these exact blocked tensors. No later-block transport is used. The complete source-u contraction in issue #5982 is a dependency of the admissible source-isometry proof, and the complete source-v contraction in issue #6071 is a dependency of the admissible simple-tensor equivalence proof. These open proof nodes do not establish the unrestricted source statements.

The same source/scoped separation is carried through the complete continuity and symmetry subtree. The continuity labels `prop:continuity-index` and `coruvSF` have the scoped counterparts `prop:mpu_admissible_continuity_index` and `cor:mpu_admissible_continuous_standard_form`. The latter is the restricted-source form of the continuous standard-form corollary in [CPGSV17, lines 867–910].

For any tensor $\mathcal{A}$, write $\mathcal{A}^\sharp$ for its physical-adjoint comparison tensor. If $\mathcal{A}_k$ is the actual k-site block, then $\mathcal{A}_k^\sharp$ denotes the physical adjoint of that blocked tensor.

For local symmetry and conjugation classification, the source labels `prop1conj`, `corconj`, `cor:sympl-to-orthogonal`, `PropChiral`, and `prop:local-char-transposition` have the scoped counterparts `prop:mpu_admissible_conjugation_symmetry`, `prop:mpu_admissible_conjugation_standard_forms`, `cor:mpu_admissible_symplectic_to_orthogonal`, `prop:mpu_admissible_local_time_reversal`, and `prop:mpu_admissible_local_transposition`. The unrestricted local characterizations reproduce the propositions themselves and therefore do not add an index-zero hypothesis; the index obstruction belongs to the surrounding source discussion. The admissible counterparts also do not assume index zero. For the supplied two-site tensor and its physical-adjoint or transposed comparison, symmetry identifies the two generated MPUs. The Admissible Fundamental Theorem therefore identifies the corresponding source spaces. The relevant operation exchanges the two source cuts, and hence

$$r(\mathcal{U}_2) = r(\mathcal{U}_2^\sharp) = \ell(\mathcal{U}_2), \tag{17}$$

with the same equations for the transposed comparison tensor. Thus $\mathrm{ind}(\mathcal{U}_2) = 0$ directly from the source ranks, and admissible blocking independence gives $\mathrm{ind}(\mathcal{U}) = 0$. The obstruction is therefore derived from the supplied source and comparison data rather than imposed as an external hypothesis. This follows the source arguments preceding Propositions IV.5 and IV.6. The source definitions `def:strictly-equivalent-symmetry` and `def:equivalent-symmetry`, and the source criterion `lem:equivalent-symmetry`, have the scoped labels `def:mpu_admissible_strict_symmetry_equivalence`, `def:mpu_admissible_symmetry_equivalence`, and `lem:mpu_admissible_symmetry_path_criterion`. The source equation at paper lines 1345–1353 repeats the symmetry-transformed operator on both sides. The blueprint retains the intended invariance equation $\mathcal{S}[W^{(N)}(p)] = W^{(N)}(p)$.

The conjugation phase results `conjphs`, `cor:all-equivalent-under-conjugation`, `prop:conjsym-strict-equiv`, and `cor:equiv-under-conjugation` have the scoped labels `thm:mpu_admissible_conjugation_odd_rank`, `cor:mpu_admissible_all_conjugation_equivale` `thm:mpu_admissible_conjugation_even_rank`, and `cor:mpu_admissible_equiv_under_conjugation`. The unrestricted all-equivalent corollary retains its source context that either r or ℓ is odd. The admissible odd-rank determinant criterion explicitly assumes reduced full-support source data for each original one-site endpoint and its conjugate comparison, in addition to the blocked path datum; these hypotheses are what define the endpoint standard-form unitaries u_i and v_i. The final admissible equivalence corollary retains both source branches. If one original source rank is odd, only Case I occurs, but no parity is inferred for the source ranks after blocking. The rank identities $r^{(2)} = dr$ and $\ell^{(2)} = d\ell$ use admissible blocking independence and therefore require simple one-site and two-site endpoints carrying their own source data. Both conjugate two-site endpoint tensors carry source data as well. When a blocked rank is odd, only the blocked product determinant is needed and the odd-rank criterion applies. When both blocked ranks are even, including the case in which an even physical dimension multiplies an odd source rank, blocking preserves Case I and parity makes the two separate blocked determinants equal to one, so the even-rank criterion applies instead. If both original ranks are even, the blocked ranks remain even and the even-rank criterion distinguishes Case I from Case II. The admissible odd-original-rank all-equivalent corollary uses this full classification rather than applying the odd-rank theorem after blocking unconditionally. Its hypotheses explicitly supply source data for the one-site and two-site endpoints and both conjugate comparisons. The admissible U_1–U_2 conjugation classification repeats these four data requirements for each of U_1, U_2, $\widetilde{U}_1$, and $\widetilde{U}_2$; an arbitrary path datum at a separate length m does not replace the endpoint data.

For the time-reversal sign, the source labels `eq:sigma-from-trace`, `lem:mpu_sigma_gauge_invariant`, `prop:sigma-well-defined`, `prop:sigma1-and-sigma2-equal`, and `cor:sigma-equivalence-necessary` have the scoped labels `lem:mpu_admissible_sigma_trace`, `lem:mpu_admissible_sigma_gauge_invariant`, `prop:mpu_admissible_sigma_well_defined`, `prop:mpu_admissible_sigma_blocking_parity`, and `cor:mpu_admissible_sigma_equivalence_necessary`. The two MPS comparison results `prop:mps-class-eq-mpu-class-for-conjugation` and `prop:dagger-mps-class-and-mpu-cla` have scoped labels `prop:mpu_admissible_mps_conjugation_class` and `prop:mpu_admissible_mps_time_rever`

Finally, the source example and swap labels `ex:all-phases-under-conjugation`, `example:czx`, `ex:dagger-all-but-czx`, `lemma:sym-trafo-swap`, `prop:U2-U3-trivial-conjugation`, `prop:U2-U3-trivial-tr-transpose`, `prop:eqiv-U1-U2-conjugate`, and `prop:U1-U2-equiv-ancillatrick` have the scoped counterparts `ex:mpu_admissible_all_conjugation_phases`, `ex:mpu_admissible_czx`, `ex:mpu_admissible_time_reversal_classes`, `lem:mpu_admissible_swap_symmetry` `prop:mpu_admissible_u2_u3_conjugation`, `prop:mpu_admissible_u2_u3_time_reversal_transposition`, `prop:mpu_admissible_u1_u2_conjugation`, and `prop:mpu_admissible_u1_u2_ancilla`. The unrestricted CZX entry defines the Hadamard matrix appearing in its tensor formula. Comment-only source locators identify the exact passages for the unitary normal-form lemma, the three topological-insulator-inspired MPUs, and their swap transformation.

Continuity requires more than endpoint data. Chapter 28 therefore introduces an admissible equivalence and path datum. It consists of a chosen strict-equivalence path, one common positive blocking length m, and continuously varying reduced full-support source data for the actual blocked path tensor $\mathcal{W}(p)_m$ at every point. The tensor $\mathcal{W}(p)_m$ itself is simple; it may not be replaced by an equivalent canonical or reduced representative. The actual blocked endpoints carry their own data. The same requirement is imposed after adjoining ancillas and blocking. This is a hypothesis attached to a chosen witness and does not assert that such a witness exists. The datum does not automatically include the source bound $m \leq D^4$. The continuous standard-form corollary assumes this bound. For a datum of length m, its standard-form families describe the actual blocked endpoint MPUs $U_1^{(mN)}$ and $U_2^{(mN)}$. They do not describe the unblocked families $U_1^{(N)}$ and $U_2^{(N)}$ unless $m = 1$. Accordingly, every admissible strict phase theorem about the original endpoint families requires a path datum of blocking length one. General admissible equivalence may still be witnessed by strict equivalence of supplied blocked endpoints.

For a path preserving a specified symmetry $\mathcal{S}$, admissibility also includes continuously varying source data for the fixed family consisting of the conjugate, physical-adjoint, transposed, and $\mathcal{S}$-image comparison tensors of the actual blocked path. Each tensor in this family is simple and carries its own datum, independently of which comparisons a later proof uses. No symmetry operation is assumed to preserve source data automatically.

With this convention, admissible equivalence implies equality of the index. The converse is claimed only when the standard-form interpolation is supplied with an admissible path datum. The admissible standard-form path criterion at length m concerns the blocked endpoint families $U_1^{(mN)}$ and $U_2^{(mN)}$. Its strict-phase descendants concern the original endpoint families only under the explicit specialization $m = 1$. In particular, no ordinary strict-equivalence or symmetry-equivalence path is silently promoted to an admissible one, and no blocked standard form is read as a standard form of an unblocked endpoint.

For a time-reversal-invariant tensor $\mathcal{U}$, let $\sigma^{(k)} \in \{\pm 1\}$ be the sign in the local time-reversal relation for a standard form of its actual k-site block $\mathcal{U}_k$.

The source proofs of the time-reversal and transposition characterizations apply the fundamental theorem after blocking two sites. Every such statement therefore supplies source data for the actual one-site endpoint tensor, its actual two-site block, and the physical-adjoint or transposed comparison tensor of that block. Defining both $\sigma^{(1)}$ and $\sigma^{(2)}$ requires source data for $\mathcal{U}$, $\mathcal{U}_2$, $\mathcal{U}_2^\sharp$, $\mathcal{U}_4$, and $\mathcal{U}_4^\sharp$. Gauge invariance of the sign is recorded separately and follows directly from the trace formula, without a path hypothesis. The blocking-parity proposition uses this gauge-invariance lemma and the explicit unwinding argument of the source; it does not invoke pathwise constancy. To compare both signs along an unblocked path $\mathcal{W}(p)$, pathwise constancy is applied separately to the continuously varying families $\mathcal{W}(p)$, $\mathcal{W}(p)_2$, $\mathcal{W}(p)_4$, $\mathcal{W}(p)_2^\sharp$, and $\mathcal{W}(p)_4^\sharp$. This is a length-one admissible path. For equivalence after adjoining ancillas, the same five continuous families are required for the actual ancilla-enlarged blocked witness path, which is then regarded as a length-one path in its own right, rather than only at its endpoints. The resulting non-strict conclusion compares only $\sigma^{(2)}$ of the two supplied witness endpoints $\mathcal{W}_{\mathrm{anc}}(0)$ and $\mathcal{W}_{\mathrm{anc}}(1)$. It does not compare $\sigma^{(2)}$ of the original MPU tensors. Promoting this witness-level equality to an original-tensor invariant requires preservation under adjoining identity ancillas, which remains open in issue #6138. All later time-reversal descendants and examples repeat the required endpoint and comparison data after blocking or adjoining ancillas. The swap transformation and its two dependent strict-phase statements require the actual source-constructed interpolation to carry an admissible symmetry-path datum of blocking length one. This datum includes the path itself and all four fixed comparison paths: conjugate, physical-adjoint, transposed, and symmetry-image. No preservation under any of these operations is asserted. In the admissible U_1–U_2 ancilla statement, the supplied length-one path begins at named ancilla-enlarged blocked endpoints $\widehat{\mathcal{U}}_1$ and $\widehat{\mathcal{U}}_2$. Only these endpoints are claimed to be admissibly strictly equivalent. No equivalence of the original ancilla-free tensors $\widetilde{U}_1$ and $\widetilde{U}_2$ is asserted. The construction adjoins equal ancillas of dimension d, which need not satisfy the coprimality condition in the blueprint definition of equivalence when $d > 1$.

For bond dimension one, the normalized transfer matrix supplies the source pair directly. For bond dimension greater than one, no current result constructs reduced full-support source data for every simple block or proves their preservation under blocking, adjoining ancillas, tensor products, composition, conjugation, adjoint, transposition, or continuous variation, or identifies the source data of a tensor with those of a reduced representative. Consequently the admissible Chapter 28 statements are restricted to the explicitly supplied tensors, comparison tensors, and chosen paths. The parallel source-labelled statements retain the hypotheses of the paper and remain unformalized. No restricted statement is presented under a source label.

5 Elimination plan

The native source-data scope tracker is issue #6136. Its four implementation branches separate the logically distinct transport problems. Issue #6137 constructs a reduced full-support representative and transports the source objects back to the tensor used in the theorem. Together with the independent complete-network issues #5982 and #6071, this is required before the admissible source-isometry and simple-tensor equivalence nodes can be merged into `lemuisometry` and `ThmFund1`. The admissible standard form can then be merged into `SF`.

Issue #6138 proves source-data preservation under physical blocking and adjoining identity ancillas. It removes the extra hypotheses from the admissible index definition, its well-definedness theorem, the blocked local symmetry statements, the sign-parity results, and the ancilla-dependent examples. Issue #6139 treats tensor products, composition, and conjugate, adjoint, transposed, and other transformed comparison tensors. It removes the extra endpoint and comparison-tensor assumptions from the admissible local symmetry and example nodes. Issue #6140 constructs continuous source data along strict-equivalence paths and their four fixed symmetry-transformed comparison paths. It removes the path hypotheses from the admissible continuity, symmetry-equivalence, phase-classification, swap, and MPS-comparison nodes.

The twenty-nine source/scoped pairs in the continuity and symmetry subtree are merged only after the relevant branches above are closed. Until then, every source label remains an unrestricted unformalized statement, while every reduced-source or path-dependent statement remains under its `mpu_admissible` label. Closing issues #6137–#6140 does not replace the independent proof obligations in issues #5982 and #6071.

6 Verdict

Verdict: scope restriction. The paper's normal-tensor proposition, the source-isometry lemma, and Theorem III.8 concern the intended reduced canonical-form-II representative, with zero-weight blocks and the ambient zero complement omitted. This reduction is not an explicit hypothesis of the normal-tensor proposition, and it does not follow from literal canonical-form membership alone.

Consequently, a source-faithful formalization must construct this reduced representative and transport the source constructions, or explicitly assume chosen canonical-form-II data with full active support. The formalization uses the latter option when $D > 1$ and the direct one-dimensional stabilization when $D = 1$. It does not yet construct the representative reduction in the higher-dimensional case. The original source labels therefore remain unrestricted and unformalized; the distinctly named admissible nodes record the current reduced full-support route.

Within Theorem III.8 itself, the two remaining proof gaps are the complete source-u contraction tracked by issue #5982 and the complete source-v equivalence tracked by issue #6071. Separately, the downstream standard-form, index, continuity, and symmetry results lack existence or preservation theorems for reduced source data under representative reduction, blocking, adjoining ancillas, tensor products, composition, transformed comparison tensors, and continuous paths. Chapter 28 treats all of these as explicit admissibility hypotheses rather than consequences of the MPU predicate. The full-support irreducibility implication is formalized by `MPSTensor.CPSVCanonicalFormData.isIrreducibleTensor_of_card_active_eq_one_of_fullActiveSupport`. The spectral-radius and primitivity conclusions are derived from the shifted normalized transfer traces and combined with full active support in `MPOTensor.IsMPU.isNormalTensor_normalizedFlattening_of_cpsv`. Bare `MPOTensor.IsMPU` is deliberately not used to infer ambient normality or an ambient positive definite fixed point.

References

[CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.