

Orientation of the Three-Swap Ancilla Circuit

2026-08-30

Verdict: false-source (resolved).

1 The printed circuit

Proposition `prop:U1-U2-equiv-ancilla` of Ref. [CPGSV17] claims that, after adjoining one identity ancilla per site, conjugating $\tilde{U}_1$ by the three swap layers in Figure `fig:TR-ancilla` gives $\tilde{U}_2$. The layers above $\tilde{U}_1$, read from the endpoint outwards, are labelled $S_{n,n+1}^{1,a}$, $S_{n,n}^{1,2}$, and $S_{n,n}^{2,a}$.

Let R denote cyclic rotation of configurations, with $(R\sigma)(n) = \sigma(n+1)$. On triples (x, y, a) of the first species, second species, and ancilla configurations, the three literal swap labels act as

$$A(x, y, a) = (Ra, y, R^{-1}x), \quad (1)$$

$$B(x, y, a) = (y, x, a), \quad (2)$$

$$D(x, y, a) = (x, a, y). \quad (3)$$

Here A is the global product of the swaps between species 1 at site n and the ancilla at site $n+1$. The first output therefore receives the next-site ancilla, while the ancilla output receives the previous-site first species.

2 The orientation mismatch

Let $C = D \circ B \circ A$ be the stack above $\tilde{U}_1$. Direct substitution into (1)–(3) gives

$$C(x, y, a) = (y, R^{-1}x, Ra), \quad (4)$$

$$C^{-1}(x, y, a) = (Ry, x, R^{-1}a). \quad (5)$$

Consequently, conjugating the physical swap gives

$$C^{-1}BC(x, y, a) = (Ry, R^{-1}x, a). \quad (6)$$

The paper defines T by $T|n_1, \dots, n_N\rangle = |n_N, n_1, \dots, n_{N-1}\rangle$. Thus T sends a ket configuration τ to $R^{-1}\tau$, and $T^\dagger$ sends it to $R\tau$. It follows that the two

swap-transformed shifts act by

$$\tilde{U}_2(x, y, a) = (R^{-1}y, Rx, a), \quad (7)$$

$$\tilde{U}_3(x, y, a) = (Ry, R^{-1}x, a). \quad (8)$$

Comparison of (6) with (7)–(8) shows that the literal circuit labels always agree with $\tilde{U}_3$. For $N \geq 3$ and $d \geq 2$, they do not in general agree with $\tilde{U}_2$: one may choose a configuration whose values at the preceding and following sites differ, so $R\sigma \neq R^{-1}\sigma$. By contrast, $R = R^{-1}$ when $N \leq 2$, and the configuration space has at most one element when $d \leq 1$. The two endpoint permutations therefore coincide in these degenerate cases.

3 Local correction and formal record

Reversing the crossed swap replaces A by

$$A_{\text{rev}}(x, y, a) = (R^{-1}a, y, Rx). \quad (9)$$

This is the swap between species 1 at site $n + 1$ and the ancilla at site n , rather than the literal $S_{n,n+1}^{1,a}$ label. Using A_{rev} in the same three-layer stack changes (6) to

$$(x, y, a) \mapsto (R^{-1}y, Rx, a), \quad (10)$$

which is the $\tilde{U}_2$ endpoint in (7). Thus reversing the orientation of the crossed labels is a local correction of the diagram; it does not alter the proposition’s intended endpoint.

The formal coordinate calculation keeps the literal printed labels and proves (6). It asserts only this permutation identity and does not identify its permutation matrix with either ancilla-enlarged MPO or use it as a dependency of the $\tilde{U}_1$ – $\tilde{U}_2$ path proposition. ¹

4 Verdict

The three swap labels in Figure **fig:TR-ancilla** agree with $\tilde{U}_3$ under the shift convention stated in the same paper. They also agree with $\tilde{U}_2$ in the degenerate cases $N \leq 2$ or $d \leq 1$, but the figure’s claimed $\tilde{U}_2$ endpoint is false as printed on nondegenerate periodic chains. Reversing the orientation of the crossed species–ancilla swaps gives the intended endpoint, so the discrepancy has a resolved local correction.

¹Formal declarations: `MPOTensor.shiftAncillaCounterShift` and `MPOTensor.permMatrix_shiftAncillaThreeSwap_endpoint` in [TNLean/MPs/MPU/Examples/ShiftSymmetryPaths](#).

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.