

The Empty-Word Summation Error in the MPS Residual Expansion

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Verdict: false-source (resolved).

1 The printed expansion

Let $A = (A^i)_i$ be a normal matrix product state tensor, let $B = (B^i)_i$ be another tensor with the same physical alphabet, and let V, W be a reduction from B to A . Thus

$$VW = \mathbf{1}, \tag{1}$$

$$VB^{i_1} \dots B^{i_n} W = A^{i_1} \dots A^{i_n} \quad (n \geq 1). \tag{2}$$

Define the residual matrices and their nilpotency length ℓ by

$$R^i = B^i - WA^iV, \tag{3}$$

$$R^{i_1} \dots R^{i_n} = 0 \quad (n \geq \ell). \tag{4}$$

The lemma immediately preceding Lemma `B.expand` proves

$$VR^{i_1} \dots R^{i_m} W = 0 \quad (m > 0). \tag{5}$$

The first equality of Lemma `B.expand` in Ref. [MGSC18] then prints

$$B^{i_1} \dots B^{i_n} = \sum_{0 \leq r \leq s \leq n} R^{i_1} \dots R^{i_r} WA^{i_{r+1}} \dots A^{i_s} VR^{i_{s+1}} \dots R^{i_n}. \tag{6}$$

The source coordinates are lines 3993–4005 of `cornerproblem.tex` in arXiv:1706.07329v2.

With the ordinary convention that an empty matrix product is the identity, the summands with $r = s$ in (6) contain WV . The sum also has no pure residual term. In particular, at $n = 1$ its right-hand side is $WVR^i + WA^iV + R^iWV$, rather than $R^i + WA^iV$.

2 A nondegenerate counterexample

Take one physical letter, a one-dimensional normal tensor A , and a three-dimensional tensor B . Set

$$A^1 = 1, \quad W = (1, 0, 0)^\top, \quad V = (1, 0, 0), \quad (7)$$

$$R^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^1 = WV + R^1. \quad (8)$$

These data satisfy

$$VW = 1, \quad (9)$$

$$V(B^1)^n W = 1 = (A^1)^n \quad (n \geq 0), \quad (10)$$

$$(R^1)^2 = 0, \quad V(R^1)^m W = 0 \quad (m > 0). \quad (11)$$

Thus all hypotheses used for (6) hold, with a nonzero nilpotent residual. Nevertheless, its $n = 1$ right-hand side is

$$WVR^1 + WA^1V + R^1WV = WV, \quad (12)$$

$$B^1 = WV + R^1 \neq WV. \quad (13)$$

Hence the printed equality is false on data satisfying the intended reduction and nilpotency conditions; the discrepancy is not a degenerate reading of a definition.

3 The corrected expansion

Expand each factor as $B^i = R^i + WA^iV$. If two selected WA^iV factors are separated by a nonempty residual word, then (5) makes that term zero. The surviving terms are therefore the pure residual word and the terms with one nonempty consecutive A -word. The corrected all-length identity is

$$B^{i_1} \dots B^{i_n} = R^{i_1} \dots R^{i_n} + \sum_{0 \leq r < s \leq n} R^{i_1} \dots R^{i_r} WA^{i_{r+1}} \dots A^{i_s} VR^{i_{s+1}} \dots R^{i_n}. \quad (14)$$

The strict inequality $r < s$ is essential: it says that the central A -word is nonempty.

The two contracted equalities printed after (6) remain valid. In the present notation they are

$$VB^{i_1} \dots B^{i_n} = \sum_{\max(0, n-\ell) \leq s \leq n} A^{i_1} \dots A^{i_s} VR^{i_{s+1}} \dots R^{i_n}, \quad (15)$$

$$B^{i_1} \dots B^{i_n} W = \sum_{0 \leq r \leq \min(\ell, n)} R^{i_1} \dots R^{i_r} WA^{i_{r+1}} \dots A^{i_n}. \quad (16)$$

For $n < \ell$, the terms with $s = 0$ and $r = n$ are the contracted pure residual word. For $n \geq \ell$, the displayed endpoints with residual length exactly ℓ vanish by (4); retaining them is harmless.

4 Verdict

The first equality in Lemma **B_expand** is false as printed because its weak summation range mishandles the empty central word. The source proof route gives the pure-residual and strict-range formula (14). The two contracted equalities remain correct with their zero endpoints understood as above.

The corrected strict-range identity and both contracted formulas are proved in the formal development under the hypotheses stated above.¹

References

- [MGSC18] András Molnár, Yimin Ge, Norbert Schuch, and J. Ignacio Cirac. A generalization of the injectivity condition for projected entangled pair states. *Journal of Mathematical Physics*, 59(2):021902, 2018. Source coordinates in this repository use arXiv:1706.07329v2.

¹Formal declarations: `MPSTensor.IsReduction.evalWord_eq_reductionResidual_add_strictWindowRangeSum`, `MPSTensor.IsReduction.mul_evalWord_eq_reductionLeftContractedRangeSum`, and `MPSTensor.IsReduction.evalWord_mul_eq_reductionRightContractedRangeSum`. The corresponding blueprint entries are `thm:mps_reduction_residual_expansion` and `thm:mps_reduction_contracted_expansions` in `blueprint/src/chapter/ch02_mps.tex`.