

The Proportionality Scalar in the MPS Reduction Theorem

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Verdict: false-source (resolved).

1 The printed statement

Let A be a normal matrix product state tensor and let B be another tensor with the same physical alphabet. Proposition 20 of Ref. [MGSC18] assumes that, for some $\lambda \in \mathbb{C}$,¹

$$V_n(B) = \lambda^n V_n(A) \quad \text{for every positive chain length } n. \quad (1)$$

It concludes that there are rectangular matrices V, W such that

$$VW = 1, \quad (2)$$

$$VB^{i_1} \dots B^{i_m} W = A^{i_1} \dots A^{i_m} \quad \text{for every word } (i_1, \dots, i_m). \quad (3)$$

The source proof subsequently says that A and B generate the same state, rather than proportional state families, and uses that equality in the construction of V, W and in the residual-algebra argument [MGSC18].²

2 Counterexamples

The printed implication is false for every nonzero $\lambda \neq 1$. Fix such a scalar and take one bond coordinate and one physical letter, with

$$A^1 = 1, \quad B^1 = \lambda. \quad (4)$$

The transfer map of A is the identity on the one-dimensional matrix algebra, and is therefore primitive. Moreover,

$$V_n(B) = \lambda^n \quad (5)$$

$$= \lambda^n V_n(A), \quad (6)$$

¹Source coordinates in `cornerproblem.tex` from arXiv:1706.07329v2: lines 3124–3133.

²The equality is asserted at `cornerproblem.tex:3839`, `cornerproblem.tex:3865`, and `cornerproblem.tex:3973--3976`; the proof of Proposition 20 occupies `cornerproblem.tex:3815--3938`.

so Eq. 1 holds. But Eq. 2 would give

$$VB^1W = \lambda VW \quad (7)$$

$$= \lambda, \quad (8)$$

whereas Eq. 3 requires $VB^1W = A^1 = 1$. Taking $\lambda = 2$ gives the simplest nondegenerate numerical instance.

The zero scalar is additionally degenerate. Taking $A^1 = 1$, $B^1 = 0$, and $\lambda = 0$ satisfies the positive-length hypothesis, but every pair obeying $VW = 1$ has

$$VB^1W = 0 \quad (9)$$

$$\neq A^1. \quad (10)$$

3 Corrected statements

The proof printed in Ref. [MGSC18] establishes the equality specialization $\lambda = 1$. If $\lambda \neq 0$, define $\tilde{B}^i = \lambda^{-1}B^i$. Then

$$V_n(\tilde{B}) = V_n(A). \quad (11)$$

Applying the equality specialization to $\tilde{B}$ gives $VW = 1$ and

$$V\tilde{B}^{i_1} \dots \tilde{B}^{i_m}W = A^{i_1} \dots A^{i_m}. \quad (12)$$

Equivalently, the corrected conclusion for the original tensor is

$$VB^{i_1} \dots B^{i_m}W = \lambda^m A^{i_1} \dots A^{i_m}. \quad (13)$$

Thus one may either assume $\lambda = 1$ and retain the printed conclusion, or assume $\lambda \neq 0$ and use Eq. 13. No corresponding conclusion follows from the printed hypotheses when $\lambda = 0$.

4 MPU specialization

The MPU application in Ref. [FRBC25] starts from the linear representation identity $U_g U_h = U_{gh}$. Hence the product tensor and the target tensor generate exactly the same operator at every positive chain length: this is the case $\lambda = 1$. The fusion equations used there are therefore unaffected by the defect in the proportional version of Proposition 20.

The formal predicate `MPSTensor.IsReduction` records the rectangular pair, the retraction $VW = 1$, and the word equation including the empty word. Scalar covariance is proved by `MPSTensor.IsReduction.smul`, and `MPSTensor.IsReduction.evalWord_smul_target` gives the corrected formula $VB^wW = \lambda^{|w|}A^w$ for a reduction from B to λA . For $\lambda \neq 0$, the injective-target and normal-target existence results are `MPSTensor.exists_isReduction_smul_of_isInjective`

`_of_mpv_eq_pow_mul` and `MPSTensor.exists_isReduction_smul_of_isNormal_of_mpv_eq_pow_mul`. They rescale the arbitrary source to $\lambda^{-1}B$, apply the equality-case existence theorem, and scale back. The corresponding Blueprint entries are `lem:mps_reduction_scalar_covariance` and `thm:mps_proportional_reduction_exists` in `blueprint/src/chapter/ch02_mps.tex`.

5 Verdict

The unscaled conclusion of Proposition 20 is false for every unrestricted choice of $\lambda \neq 1$, not merely in the degenerate zero case. The source proof establishes the $\lambda = 1$ specialization. For nonzero λ , the corrected general word relation is Eq. 13; the zero case requires separate hypotheses. The MPU use is source-faithful because it supplies exact equality.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.
- [MGSC18] András Molnár, Yimin Ge, Norbert Schuch, and J. Ignacio Cirac. A generalization of the injectivity condition for projected entangled pair states. *Journal of Mathematical Physics*, 59(2):021902, 2018. Source coordinates in this repository use arXiv:1706.07329v2.