

Nilpotency Length One after Blocking: A Terminology Offset

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Verdict: clarification (resolved).

1 The two statements

Let $A = (A^i)_i$ be an injective matrix product state tensor, let $B = (B^i)_i$ be a tensor with the same physical alphabet and the same positive-length closed-chain coefficients, and let (V, W) be a reduction from B to A , so that $VW = \mathbf{1}$ and $VB^{i_1} \dots B^{i_n} W = A^{i_1} \dots A^{i_n}$ for every word. Write $N^i = B^i - WA^iV$ for the residual letters. Definition 8 of Ref. [MGSC18] calls the least N_0 with

$$N^{i_1} \dots N^{i_n} = 0 \quad (n \geq N_0) \quad (1)$$

the nilpotency length of the selected reduction.¹ Thus $N_0 = 1$ means $N^i = 0$ for every letter, that is, $B^i = WA^iV$.

Ref. [FRBC25] states the exterior fusion identity

$$B^{\mathbf{p}}WA^{\mathbf{c}}VB^{\mathbf{q}} = B^{\mathbf{p}\mathbf{c}\mathbf{q}} \quad (2)$$

for all nonempty central words $\mathbf{c}$ and all exterior words with $|\mathbf{p}|, |\mathbf{q}| \geq \ell$, “for some ℓ called the nilpotency length”, and then says that “ ℓ can always be reduced to 1 by blocking the MPUs a finite number of times”.² The footnote to Theorem 1 of Ref. [GRS24] makes the same assertion: the theorem “assumes the nilpotency length of the off-diagonal blocks of B is 1”, which “can always be achieved by blocking”.³

2 The offset

The corrected residual expansion [con26] gives (2) as soon as

$$N_0 \leq |\mathbf{p}| + 1, \quad N_0 \leq |\mathbf{q}| + 1, \quad (3)$$

¹Source coordinates in `cornerproblem.tex` from arXiv:1706.07329v2: lines 3147–3152.

²`main.tex` line 1498 of arXiv:2502.20257; the fusion equations occupy lines 1409–1498.

³`MPU-DW.tex` lines 358–364 of arXiv:2405.00439v2.

because the adjacent central letter contributes one residual factor to each exterior residual word. The bound (3) is sufficient, not necessary, so it exhibits $N_0 - 1$ as *an* exterior buffer length: the least exterior buffer length ℓ in the sense of Ref. [FRBC25] satisfies $\ell \leq N_0 - 1$ for $N_0 \geq 1$. Correspondingly, $N_0 \leq 2$ implies $\ell \leq 1$, that is, one exterior site suffices. The sources write this as “ $\ell = 1$ ”, which is the sufficiency of one exterior site and not an exact value: for $B = A$ with the identity reduction the residuals vanish, so $N_0 = 1$ while the exterior identity already holds with empty buffers and $\ell = 0$.

Blocking L original sites into one blocked site turns (3) into the condition $N_0 \leq L + 1$ for one blocked exterior site. For a finite family of reductions with nilpotency lengths $N_0(g, h)$, the length

$$L = \max(1, \max_{(g,h)} (N_0(g, h) - 1)) \quad (4)$$

is positive and satisfies $N_0(g, h) \leq L + 1$ for every pair. After this common block, (2) holds for every pair with $|\mathbf{p}|, |\mathbf{q}| \geq 1$ blocked sites. This is the content of the two quoted sentences that the formal development uses.

Blocking does not, in general, make the blocked residual letters vanish, and the blocked nilpotency length of Definition 8 need not be 1. The two readings differ on nondegenerate data.

3 A counterexample to the literal reading

Take the physical letters $i \in \{0, 1\}$, the one-dimensional injective tensor $A^0 = A^1 = 1$, and the two-dimensional tensor

$$B^i = \begin{pmatrix} 1 & x_i \\ 0 & 0 \end{pmatrix}, \quad x_0 = 0, \quad x_1 = 1. \quad (5)$$

Every word evaluates to $B^{i_1} \dots B^{i_n} = B^{i_n}$, so all positive-length closed-chain coefficients equal $1 = A^{i_1} \dots A^{i_n}$. With $V = (1, 0)$ and $W = (1, 0)^\top$,

$$VW = 1, \quad VB^{i_1} \dots B^{i_n}W = 1, \quad N^i = \begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix}, \quad (6)$$

so (V, W) is a reduction, $N^1 \neq 0$, and $N^i N^j = 0$. Its nilpotency length is $N_0 = 2$.

Block $L \geq 1$ sites. The blocked letter for the word $\mathbf{w} = (i_1, \dots, i_L)$ is $B^{\mathbf{w}} = B^{i_L}$, which depends on the last original letter, while the blocked target letter is $A^{\mathbf{w}} = 1$. For any reduction (V', W') of the blocked tensors, the blocked residual letter is $B^{i_L} - W'V'$, and the two values $B^0 - W'V'$ and $B^1 - W'V'$ cannot both vanish. Thus no reduction of any blocking has nilpotency length 1 in the sense of Definition 8.

One exterior site nevertheless suffices. For $|\mathbf{q}| \geq 1$,

$$B^{\mathbf{p}} W A^{\mathbf{c}} V B^{\mathbf{q}} = B^{\mathbf{p}} W V B^{\mathbf{q}} = B^{i_{|\mathbf{q}|}} = B^{\mathbf{p}^{\mathbf{c}} \mathbf{q}}, \quad (7)$$

since $B^{\mathbf{p}}WV = B^0$ for every $\mathbf{p}$, and $B^0B^{\mathbf{q}} = B^{\mathbf{q}}$ is the last letter of $\mathbf{q}$. The right buffer is necessary: for empty $\mathbf{q}$ and $\mathbf{c} = (1)$, the left side is $B^{\mathbf{p}}WV = B^0 \neq B^1 = B^{\mathbf{p}\mathbf{c}}$. This agrees with (3) at $N_0 = 2$.

4 Convention

The formal development reads “the nilpotency length is reduced to 1 by blocking” as the common-block statement of Section 2: one positive blocking length with $N_0(g, h) \leq L + 1$ for every pair, after which one blocked exterior site satisfies the exterior fusion identity. It does not require a blocked residual letter to vanish and does not assert that the blocked nilpotency length equals 1.⁴

References

- [con26] The TNLean contributors. The empty-word summation error in the MPS residual expansion. Paper-gap note mgsc18_residual_expansion_empty_word_error, TNLean, 2026. http://sirui-lu.com/TNLean/paper-gaps/mgsc18_residual_expansion_empty_word_error.pdf.
- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.
- [GRS24] José Garre-Rubio and Norbert Schuch. Fractional domain wall statistics in spin chains with anomalous symmetries. arXiv:2405.00439v2, 2024.
- [MGSC18] András Molnár, Yimin Ge, Norbert Schuch, and J. Ignacio Cirac. A generalization of the injectivity condition for projected entangled pair states. *Journal of Mathematical Physics*, 59(2):021902, 2018. Source coordinates in this repository use arXiv:1706.07329v2.

⁴Formal declarations: `MPSTensor.IsReductionExteriorBufferLength`, `MPSTensor.IsReduction.blockTensor_commonBufferLength_one`, `MPOTensor.GroupFamily.ReductionFamily.block_hasExteriorBufferLength_one`, and `MPOTensor.GroupFamily.IsRepresentation.exists_block_reductionFamily_one`. The corresponding blueprint entries are `thm:mps_reduction_common_blocking` in `blueprint/src/chapter/ch02_mps.tex` and `thm:mpug_common_blocking_one_site` in `blueprint/src/chapter/ch29_mpu_gauging.tex`.