

CZY fusion: associativity on the physical image

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Verdict: false-source (resolved).

1 The printed assertion

Section II of arXiv:2509.03600v1 introduces tensors A_0, A_1 of bond dimensions three and two and fusion maps $Y_{a,b}$, with $a, b \in \mathbb{Z}_2$. The maps are specified in Appendix B as the first rows of the printed changes of basis. The source then asserts

$$Y_{a+b,c}(Y_{a,b} \otimes I_c) = \omega(a, b, c)Y_{a,b+c}(I_a \otimes Y_{b,c}), \quad (1)$$

where $\omega(1, 1, 1) = -1$ and $\omega = 1$ otherwise. The local source is **references/2509.03600/main.tex**, lines 283–294 and 664–716. Product coordinates are ordered lexicographically.

The tensors and fusion maps satisfy the printed intertwining identities. Moreover, (1) is valid at $(1, 1, 1)$, with the printed normalization $Y_{1,1} = X_{1,1}[1:3, :]$ and its factor $1/\sqrt{2}$. These facts are proved independently of the unrestricted associativity assertion.

2 An obstruction on the full virtual space

At the neutral triple the relevant map is

$$Y_{0,0} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \end{pmatrix}. \quad (2)$$

Let $L = Y_{0,0}(Y_{0,0} \otimes I_3)$ and $R = Y_{0,0}(I_3 \otimes Y_{0,0})$, both acting on $(\mathbb{C}^3)^{\otimes 3}$. Using indices 1, 2, 3, the column $(3, 3, 1)$ is the twenty-fifth column. Direct multiplication gives $L_{1,(3,3,1)} = 2$ and $R_{1,(3,3,1)} = 0$. Consequently $L \neq zR$

for every $z \in \mathbb{C}$. This is an obstruction to (1) for the full virtual maps, not merely a normalization of its sign. The kernel proof is `CZYCompression.czyFusion_printed_neutral_no_scalar`.

3 The corrected statement

Two changes suffice. First, choose the normalization

$$F_{0,0} = Y_{0,0}, \quad F_{0,1} = 2Y_{0,1}, \quad F_{1,0} = 2Y_{1,0}, \quad F_{1,1} = 2\sqrt{2}Y_{1,1}. \quad (3)$$

Equivalently, after removing a common factor two, these maps are $Y_{0,0}/2$, $Y_{0,1}$, $Y_{1,0}$ and $\sqrt{2}Y_{1,1}$. The extra normalization of $Y_{1,1}$ cancels from every triple comparison. It makes the chosen maps integral. All one-sided intertwining identities remain valid under these scalar changes.

Second, restrict the equality to the physical images. Define

$$T_{a,b,c}^{ik} = \sum_{j,l=0}^1 A_a^{ij} \otimes A_b^{jl} \otimes A_c^{lk}, \quad L_{a,b,c} = F_{a+b,c}(F_{a,b} \otimes I_c), \quad R_{a,b,c} = F_{a,b+c}(I_a \otimes F_{b,c}). \quad (4)$$

Both bond orders are identified with the ordered triple (a, b, c) . The corrected identity is

$$L_{a,b,c} T_{a,b,c}^{ik} = (-1)^{abc} R_{a,b,c} T_{a,b,c}^{ik}. \quad (5)$$

It holds at all eight group triples and all physical index pairs. The formal statements are `CZYCompression.czyFusion_supported_associativity_int` and `CZYCompression.czyFusion_supported_associativity`.

For a nonempty physical word $w = (p_1, \dots, p_n)$, multiplication of (5) on the right gives $L_{a,b,c} T^{p_1} \dots T^{p_n} = (-1)^{abc} R_{a,b,c} T^{p_1} \dots T^{p_n}$. This is `CZYCompression.czyFusion_supported_associativity_word`. Thus the phase relation holds on $\mathcal{S}_{a,b,c} = \sum_{i,k} \text{ran } T_{a,b,c}^{ik}$, and on the images of all nonempty word products.

The correction keeps the redundant virtual directions that make A_0 non-semisimple, while specifying the subspace on which a scalar associator is meaningful. It does not identify the raw fusion maps on the entire virtual space. Nor does it, by itself, provide a general definition of fusion associators for nonsemisimple tensor families. Those require a specified quotient of morphisms or invariant physical subspaces.

4 Status

The printed full-space assertion is false. Its nontrivial-triple identity and the corrected physical-image relation are proved, without assuming an associator relation. The blueprint treats them as separate statements. The mathematical follow-up is tracked in [GitHub issue #8425](#).