The Finite-Range Cyclic Knabe Coefficient

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Verdict: clarification (resolved).

1 Source statements

For a nearest-neighbor chain, let m be the number of local interaction terms in each window and let γ be the gap of the corresponding open window Hamiltonian. Knabe's finite-size criterion gives the threshold $\gamma > 1/m$. This threshold originates in [Kna88]. It is quoted for MPS parent Hamiltonians in lines 1483–1489 of the local source for [PGVWC07] and in lines 2194–2197 of the local source for [CPGSV21]. The quoted passages state only the threshold. The closed-form periodic coefficient

$$\delta = \frac{m\gamma - 1}{m - 1} \tag{1}$$

is obtained by the present derivation, as the range-two case of the coefficient below.

2 Finite-range derivation

Let h_i, $i \in \mathbb{Z}/N\mathbb{Z}$, be orthogonal projections, let $H = \sum_i h_i$, and define

$$A_s = \sum_{a=0}^{m-1} h_{s+a}, \qquad Q_q = \sum_i \big(h_i h_{i+q} + h_i h_{i-q}\big). \tag{2}$$

Assume $1 \le R \le m$, $N \ge 2m$, and that terms at cyclic separation at least R commute. Cyclic double counting gives

$$\sum_s A_s^2 = mH + \sum_{q=1}^{m-1} (m - q) Q_q. \tag{3}$$

The projection inequality $h_i h_j + h_j h_i \le h_i + h_j$ gives $Q_q \le 2H$. The separated terms satisfy $Q_q \ge 0$ for $q \ge R$, and expansion of H^2 gives

$$H + \sum_{q=1}^{m-1} Q_q \le H^2. \tag{4}$$

Set $c = m - R + 1$, so that $m - q = c + (R - 1 - q)$. The correction term splits as

$$\sum_{q=1}^{m-1}(R-1-q)Q_q = \sum_{q=1}^{R-2}(R-1-q)Q_q + \sum_{q=R}^{m-1}(R-1-q)Q_q, \tag{5}$$

$$\sum_{q=1}^{R-2}(R-1-q)Q_q \le 2H\sum_{q=1}^{R-2}(R-1-q), \tag{6}$$

$$\sum_{q=R}^{m-1}(R-1-q)Q_q \le 0. \tag{7}$$

Indeed, the first sum has positive weights and $Q_q \le 2H$ termwise. In the second sum, $R - 1 - q \le 0$ and $Q_q \ge 0$: the latter estimate applies because $R \le q \le m - 1 \le N - R$. The triangular sum is

$$\sum_{q=1}^{R-2}(R-1-q) = \frac{(R-1)(R-2)}{2}. \tag{8}$$

Therefore

$$\sum_s A_s^2 \le cH^2 + \big(m - c + (R-1)(R-2)\big)H, \tag{9}$$

$$m - c + (R-1)(R-2) = (R-1)^2, \tag{10}$$

$$\sum_s A_s^2 \le cH^2 + (R-1)^2 H. \tag{11}$$

Consequently, the local inequalities $A_s^2 \ge \gamma A_s$ imply

$$H^2 \ge \frac{m\gamma - (R-1)^2}{m - R + 1} H. \tag{12}$$

The condition $(R-1)^2 < m\gamma$ makes this coefficient positive. At $R = 2$ it reduces to Knabe's nearest-neighbor coefficient.

3 Verdict

The nearest-neighbor threshold is source-attributed as above. The coefficient for arbitrary finite range is a TNLean derivation, formalized by the cyclic projection theorem.[1] It is not a statement attributed to Knabe, to the gap-from-hypotheses criterion [Nac96, Theorem 2.1(i)] of Nachtergaele's martingale argument, or to the two MPS reviews. The abstract result concerns a cyclic family of projections. Applying it to every sufficiently large MPS parent Hamiltonian still requires a uniform gap for the corresponding nonwrapping open interaction.

[1] Formal declaration: `ProjectionGeometry.quadraticForm_sum_projections_of_cyclic_knabe`.

References

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.

[Kna88] Stefan Knabe. Energy gaps and elementary excitations for certain VBS-quantum antiferromagnets. *Journal of Statistical Physics*, 52(3–4):627–638, 1988.

[Nac96] Bruno Nachtergaele. The spectral gap for some spin chains with discrete symmetry breaking. *Communications in Mathematical Physics*, 175(3):565–606, 1996.

[PGVWC07] D. Pérez-García, F. Verstraete, M. M. Wolf, and J. I. Cirac. Matrix product state representations. *Quantum Information & Computation*, 7(5):401–430, 2007.