

The Product-Marginal Reference in the Relative-Entropy Form of Strong Subadditivity

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Verdict: clarification (resolved).

1 The source identity

Hayden, Jozsa, Petz, and Winter first give the bipartite identity

$$D(\omega_{XY} \parallel \omega_X \otimes \omega_Y) = S(\omega_X) + S(\omega_Y) - S(\omega_{XY}) \quad (1)$$

in their equation (4). Its specialization to the bipartition $A|BC$ is their equation (5). Subtracting the corresponding identity for $A|B$ gives their equation (6), namely

$$\begin{aligned} D(\rho_{ABC} \parallel \rho_A \otimes \rho_{BC}) - D(\rho_{AB} \parallel \rho_A \otimes \rho_B) \\ = S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B). \end{aligned} \quad (2)$$

Here the second pair is the image of the first under $T = \text{tr}_C$. Consequently, equality in strong subadditivity is equivalent to equality in relative-entropy data processing for this pair. The paragraph following equation (7) identifies precisely this equivalence. All these equations occur on page 3 of arXiv:quant-ph/0304007v2.

The statement has no strict-positivity assumption. If one of the marginals is singular, the relative entropy is interpreted on its support. The support of a bipartite state is contained in the tensor product of the supports of its two marginals, so both relative entropies in (2) are finite.

2 The equivalent maximally mixed criterion

The current relative-entropy theory proves the corresponding identity with a maximally mixed state on A :

$$\begin{aligned} D\left(\rho_{ABC} \left\| \frac{\mathbf{1}_A}{d_A} \otimes \rho_{BC}\right.\right) - D\left(\rho_{AB} \left\| \frac{\mathbf{1}_A}{d_A} \otimes \rho_B\right.\right) \\ = S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B). \end{aligned} \quad (3)$$

Indeed, each relative entropy is respectively $\log d_A + S(\rho_{BC}) - S(\rho_{ABC})$ and $\log d_A + S(\rho_B) - S(\rho_{AB})$. The common logarithm cancels. Thus (3) gives the same equality criterion as (2), without adding a hypothesis to the state.

This is formalized by the maximally mixed-reference equality theorem.

¹ The partial-trace identity verifies that the reference on $A \otimes B$ is the image of the reference on $A \otimes B \otimes C$. ²

This theorem is a hypothesis-free equivalent formulation with a different reference state. The exact Hayden–Jozsa–Petz–Winter formulation is now also proved.

3 The singular product-marginal evaluation

The relative entropy in the development is the total trace-log expression

$$D(\rho \parallel \sigma) = \text{Re tr}(\rho(\log \rho - \log \sigma)). \quad (4)$$

For positive definite tensor factors the logarithm of a tensor product splits. For singular positive semidefinite factors A and B , let P_A and P_B be their support projections. The support form of the tensor logarithm is

$$\log(A \otimes B) = (\log A) \otimes P_B + P_A \otimes (\log B). \quad (5)$$

Here the logarithm is extended by zero on the kernel, as in the total trace-log expression above. Indeed, simultaneous diagonalization of the two tensor factors reduces the claim to

$$\log(ab) = \log(a)\mathbf{1}_{b \neq 0} + \mathbf{1}_{a \neq 0}\log(b), \quad (6)$$

¹Formal declaration: `isSSAEquality_iff_maximally_mixed_reference_data_processing_eq` in `TNLean/Channel/Schwarz/SSAEqualityDPI.lean`.

²Formal declaration: `SSAPosDef.traceC_ABC_kronecker_traceA_ABC` in `TNLean/Channel/Schwarz/StrongSubadditivityPosDef.lean`.

for nonnegative eigenvalues a, b . This is the ordinary logarithm identity when both eigenvalues are positive, while both sides vanish when either eigenvalue is zero. This spectral calculation is an analytic justification for the singular form of the product-marginal evaluation in equation (4) of Hayden–Jozsa–Petz–Winter; the tensor-logarithm identity is not stated verbatim there. It is formalized by `Matrix.log_kronecker_posSemidef`.

The resulting hypothesis-free bipartite identity is

$$D(\omega_{XY} \parallel \omega_X \otimes \omega_Y) = S(\omega_X) + S(\omega_Y) - S(\omega_{XY}) \quad (7)$$

for every positive semidefinite ω_{XY} . The lifted support projections $P_X \otimes \mathbf{1}_Y$ and $\mathbf{1}_X \otimes P_Y$ both fix ω_{XY} . Pairing (5) with ω_{XY} therefore removes the support projections, and the two partial-trace adjoint identities give (7). This is formalized by `quantumRelativeEntropy_product_marginals`. It is precisely equation (4) of Hayden–Jozsa–Petz–Winter.

Applying (7) to the bipartitions $A|BC$ and $A|B$ gives equations (5) and (6), hence proves (2). Elementary rearrangement then gives the exact biconditional with equality in data processing. This is formalized by the declaration

`isSSAEquality_iff_product_marginal_data_processing_eq`.

No recovery map or Petz equality theorem is needed for this step.

4 Downstream recovery corollary

The equality criterion above now feeds the singular-support Petz theorem. The automatic inclusion

$$\ker(\rho_A \otimes \rho_{BC}) \subseteq \ker \rho_{ABC} \quad (8)$$

is formalized by `Matrix.product_marginal_support`. Consequently, strong-subadditivity equality implies

$$\mathcal{R}_{\rho_A \otimes \rho_{BC}}^{\text{raw}}(\rho_{AB}) = \rho_{ABC}, \quad (9)$$

$$(\text{id}_A \otimes \widehat{\mathcal{R}}_{\rho_{BC}})(\rho_{AB}) = \rho_{ABC}. \quad (10)$$

The declarations `Matrix.partialTraceRightPetzMap_product_marginal_recovery_of_isSSAEquality` and `Matrix.idTensor_partialTraceRightPetzChannel_traceC_ABC_eq_of_isSSAEquality` formalize HJPW Theorem 3 and equations (8), (10), and (11) at this exact product reference. The identity-tensored statement is supported-input only; it does not assert a global factorization of the generic completed singular product-reference channel.

5 Verdict

The gap is closed. Equality in strong subadditivity is identified with saturation of partial-trace data processing for the exact product-marginal pair of Hayden–Jozsa–Petz–Winter on the full density-matrix domain. The maximally mixed-reference theorem remains as a separate equivalent criterion.

References

- [1] P. Hayden, R. Jozsa, D. Petz, and A. Winter, *Structure of states which satisfy strong subadditivity of quantum entropy with equality*, Communications in Mathematical Physics **246** (2004), 359–374; arXiv:quant-ph/0304007v2.