# Support scope of the HJPW product-reference Petz factorization

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**Verdict:** scope-restriction (resolved).

## 1 Source statement

Let

$$T = \mathrm{id}_A \otimes \mathrm{tr}_C, \qquad \sigma_{ABC} = \rho_A \otimes \rho_{BC}. \tag{1}$$

Hayden–Jozsa–Petz–Winter write the recovery operation as

$$\widehat{T} = \mathrm{id}_A \otimes \widehat{R} \tag{2}$$

in equation (10), where $\widehat{R}$ is associated with $\rho_{BC}$ and $\mathrm{tr}_C$. Their Petz formula is specified on the support of $T\sigma$. Thus equation (2) is literal on the reduced support spaces. On the full ambient matrix algebras it requires a choice of extension away from those supports.

## 2 Global raw formula

Let $P_A$ be the support projection of $\rho_A$. The product reference and its retained marginal satisfy

$$\sigma_{ABC} = \rho_A \otimes \rho_{BC}, \qquad \mathrm{tr}_C\, \sigma_{ABC} = \rho_A \otimes \rho_B, \qquad \rho_B = \mathrm{tr}_C\, \rho_{BC}. \tag{3}$$

For positive semidefinite factors, square roots, support inverse square roots, and support projections factor across the tensor product. Consequently the raw support-Petz map on the full ambient algebra is

$$\mathcal{R}^{\mathrm{raw}}_{\rho_A \otimes \rho_{BC}} = \mathcal{S}_{P_A} \otimes \mathcal{R}^{\mathrm{raw}}_{\rho_{BC}}, \qquad \mathcal{S}_{P_A}(X) = P_A X P_A. \tag{4}$$

Equivalently,

$$\mathcal{R}^{\mathrm{raw}}_{\rho_A \otimes \rho_{BC}}(A_0 \otimes X_B) = (P_A A_0 P_A) \otimes \mathcal{R}^{\mathrm{raw}}_{\rho_{BC}}(X_B). \tag{5}$$

All identities use the canonical reassociation $(A \times B) \times C \simeq A \times (B \times C)$.

The corresponding declarations are

`Matrix.productTensorReference`
`Matrix.productTensorReference_posSemidef`
`Matrix.partialTraceRight_productTensorReference`
`Matrix.cfcSqrt_productTensorReference`
`Matrix.supportInvSqrt_partialTraceRight_productTensorReference`
`Matrix.supportProj_partialTraceRight_productTensorReference`
`Matrix.supportCompressionMap`
`Matrix.partialTraceRightPetzMap_productTensor_kronecker`
`Matrix.partialTraceRightPetzMap_productTensor`

in `TNLean/Channel/PetzProductReference`.

# 3 Identity-tensored corollaries and extension boundary

If an input $X$ is supported on the first factor,

$$(P_A \otimes \mathbf{1}_B)X(P_A \otimes \mathbf{1}_B) = X, \tag{6}$$

then support compression disappears and

$$\mathcal{R}^{\mathrm{raw}}_{\rho_A \otimes \rho_{BC}}(X) = (\mathrm{id}_A \otimes \mathcal{R}^{\mathrm{raw}}_{\rho_{BC}})(X). \tag{7}$$

If $\rho_A$ is positive definite, then $P_A = \mathbf{1}_A$, so this identity holds globally. These conclusions are formalized by

`Matrix.tensorMapId_supportCompressionMap`
`Matrix.partialTraceRightPetzMap_productTensor_apply_of_left_supported`
`Matrix.partialTraceRightPetzMap_productTensor_of_posDef_left`.

If $\rho_A$ is singular, the ambient raw map is not globally $\mathrm{id}_A \otimes \mathcal{R}^{\mathrm{raw}}_{\rho_{BC}}$. For example, an operator supported in the kernel of $\rho_A$ is killed by $X \mapsto P_A X P_A$ but not by the identity map. A global channel of the form $\mathrm{id}_A \otimes \widehat{\mathcal{R}}_{\rho_{BC}}$ is therefore a chosen tensor-compatible extension of the support formula.

TNLean's generic completed channel uses the complementary projection

$$\mathbf{1}_{AB} - P_A \otimes P_B. \tag{8}$$

For singular $\rho_A$ this projection does not generally factor as the identity on $A$ tensored with a projection on $B$. No factorization of that generic completed channel is claimed. The existing maximally mixed completed factorization is valid because the first-factor support is full.

# 4 SSA product-reference specialization

For every positive semidefinite bipartite operator $\omega_{XY}$,

$$\ker(\omega_X \otimes \omega_Y) \subseteq \ker \omega_{XY}. \tag{9}$$

Thus the product-marginal reference automatically satisfies the support condition in HJPW Theorem 3. If a tripartite density matrix attains equality in strong subadditivity, the raw product-reference Petz map recovers it:

$$\mathcal{R}^{\mathrm{raw}}_{\rho_A\otimes\rho_{BC}}(\rho_{AB}) = \rho_{ABC}. \tag{10}$$

Since $\rho_{AB}$ is supported by both $P_A\otimes\mathbf{1}_B$ and $\mathbf{1}_A\otimes P_B$, the supported factorization and the local trace-preserving completion give

$$(\mathrm{id}_A\otimes\mathcal{R}^{\mathrm{raw}}_{\rho_{BC}})(\rho_{AB}) = (\mathrm{id}_A\otimes\widehat{\mathcal{R}}_{\rho_{BC}})(\rho_{AB}) = \rho_{ABC}. \tag{11}$$

These are HJPW equations (10)–(11) on the supported input. They are formalized in `TNLean/Channel/Schwarz/SSAEqualityPetzRecovery` by

`Matrix.product_marginal_support`
`Matrix.partialTraceRightPetzMap_product_marginal_recovery_of_isSSAEquality`
`Matrix.idTensor_partialTraceRightPetzMap_traceC_ABC_eq_of_isSSAEquality`
`Matrix.partialTraceRightPetzChannel_traceA_ABC_isKrausCPTP`
`Matrix.idTensor_partialTraceRightPetzChannel_traceC_ABC_eq_of_isSSAEquality`.

No global identity-tensored factorization of TNLean's generic completed singular product-reference channel is claimed.

# 5 Structural implication complete

The product-reference recovery identity, the invariant conditional family, the joint-support block theorem, and the ambient extension now give

$$H_B \cong \bigoplus_j H_{B_j^L}\otimes H_{B_j^R}, \tag{12}$$

$$W^\dagger\rho_{ABC}W = 0_\perp\oplus\bigoplus_j \omega_j\otimes\widehat{\rho}_j, \qquad W = \mathbf{1}_A\otimes(U_B\otimes\mathbf{1}_C). \tag{13}$$

The supported matrices $\widehat{\rho}_j$ are the trace-one sector output states. The complementary blocks are zero. The forward characterization sets $p_j = \operatorname{Re}\operatorname{tr}\omega_j$, proves $p_j\geq 0$ and $\sum_j p_j = 1$, reverses each middle fibre to the order $B_j^L\otimes B_j^R$, and chooses the Hayashi unitary $U_B^\dagger$. Every supported right factor remains its sector output state. Total positive-semidefinite normalization supplies the left density matrices; at zero weight it may choose a left filler, and on a complementary sector it may choose fillers on both sides. Their weighted contributions vanish.

This completes the structural implication and is formalized by `Matrix.hayashi_ssa_equality_characterization_forward`, with the public root name `hayashi_ssa_equality_characterization_forward`.

The singular-support scope remains unchanged. The raw support-Petz map factors as in (4), and the identity-tensored formula holds on supported inputs.

For the SSA marginal, the chosen completed local channel agrees with the raw map on that supported input. No global identity-tensored factorization is claimed for TNLean's generic completed singular product-reference channel away from support.

# 6 Common invariant algebra: formalized under an explicit full-support hypothesis

A full-support specialization of a slice of the operator-algebraic Koashi–Imoto appendix (HJPW, Appendix A, lines 755–882) is now formalized independently of the product-reference recovery step above. For a finite, nonempty family of states $\rho_1, \ldots, \rho_K$, the set

$$\mathbf{F} = \left\{ F : \forall k \ F\rho_k = \rho_k \right\} \tag{14}$$

of trace-preserving completely positive operations leaving every $\rho_k$ invariant (lines 825–827) is formalized as a bundled Kraus-family type, and the common invariant algebra

$$A_0 = \bigcap_{F \in \mathbf{F}} A_F, \qquad A_F = \{X : F^*(X) = X\}, \tag{15}$$

(line 843) is formalized as the intersection, over every such $F$, of the fixed-point $*$-subalgebra of the adjoint map – together with its membership characterization and the generic block form of a star-subalgebra specialized to $A_0$. Because all dimensions are finite, $A_0$ is additionally realized as a finite intersection $A_0 = A_{F_1} \cap \cdots \cap A_{F_M}$, and by a single preserving operation $F_0 = (1/M) \sum_{\mu=1}^{M} F_\mu$ with $A_0 = A_{F_0}$ (lines 844–849). The trace-pairing adjoint of the Schrödinger mean-ergodic projection for $F_0$ is formalized as $P_0^*$. It is positive, unital, and idempotent, with range exactly $A_0$. HJPW writes this projection as the Cesàro limit of the Heisenberg iterates (lines 838–851), and those adjoint averages converge to the trace-adjoint projection. The corresponding declarations are

```
Kraus.commonAverage
Kraus.IsPreserving
Kraus.PreservingKrausFamily
Kraus.map_commonAverage_of_isPreserving
Kraus.commonInvariantStarSubalgebra
Kraus.mem_commonInvariantStarSubalgebra
Kraus.exists_unitary_conj_blockDiagonal_iff_commonInvariantStarSubalgebra
Kraus.exists_finset_preservingKrausFamily_iInf_eq
Kraus.averagedPreservingKrausFamily
Kraus.exists_preservingKrausFamily_adjointFixedPointsStarSubalgebra_eq
Kraus.commonInvariantMeanErgodicProjection
Kraus.tendsto_birkhoffAverage_commonInvariantMeanErgodicProjection
Kraus.commonInvariantMeanErgodicProjection_isPositiveMap
Kraus.commonInvariantMeanErgodicProjection_one
```

```
Kraus.commonInvariantMeanErgodicProjection_idempotent
Kraus.mem_range_commonInvariantMeanErgodicProjection_iff
Kraus.exists_commonInvariant_fixedPointBlockForm
Kraus.exists_commonInvariant_normalizedStateBlockForm
Kraus.exists_commonInvariant_normalizedStateBlockForm_preservingBlockAct
ion
```

in `TNLean/Channel/KoashiImoto/CommonInvariantAlgebra`, `TNLean/Channel/KoashiImoto/FiniteRealization`, `TNLean/Channel/KoashiImoto/PooledKrausFamily`, `TNLean/Channel/KoashiImoto/SingleWitness`, `TNLean/Channel/KoashiImoto/MeanErgodicProjection`, and `TNLean/Channel/KoashiImoto/PreservingBlockAction`. The fixed-point block-form declaration, in `TNLean/Channel/KoashiImoto/FamilyFixedPointBlockForm`, specializes the full-support density-block classification to $F_0$: one unitary direct-sum tensor decomposition and one family of density factors $\sigma_j$ work simultaneously for all $\rho_k$,

$$U^\dagger \rho_k U = \bigoplus_j \sigma_j \otimes X_{j,k}. \tag{16}$$

The declaration in `TNLean/Channel/KoashiImoto/NormalizedFamilyBlockForm` assumes that the family members are density matrices, proves that the $X_{j,k}$ are positive semidefinite, and normalizes them as

$$X_{j,k} = q_{j|k}\tau_{j|k}, \qquad q_{j|k} \geq 0, \qquad \sum_j q_{j|k} = 1, \tag{17}$$

where each $\tau_{j|k}$ is a density matrix. Thus the full-support specialization of HJPW Property 1, lines 785–789 and 857–858, is formalized, with the two tensor factors written in the reverse order. Its unrestricted joint-support form is supplied by the joint-support declaration below.

The preserving-block-action declaration proves the full-support specialization of HJPW Property 2′, lines 808–816 and 860–882, in the same coordinates. For every preserving operation $F$ and every block $j$, it produces a local trace-preserving completely positive operation $F_j$ such that

$$F_j(\sigma_j) = \sigma_j, \tag{18}$$

$$\widetilde{F}\big(\iota_j(A \otimes B)\big) = \iota_j\big(F_j(A) \otimes B\big). \tag{19}$$

Here the factor order is reversed from HJPW, so the local operation acts on the first factor. This statement concerns diagonal summands; it does not claim the Stinespring form of Property 2 or an action on off-diagonal coherences.

HJPW's argument reduces to the case where the common average $(1/K)\sum_k \rho_k$ has full support, by shrinking the working Hilbert space to the joint support of the $\rho_k$ (lines 761–763). The declaration `Kraus.exists_commonAverageSupportCompression` now constructs this support coordinate space, proves that the compressed average is positive definite, reconstructs every family member, and restricts every preserving operation to a trace-preserving operation on the support. It also proves the intertwining identity that transports the compressed channel action back through the support isometry. The declaration `Kraus.exists_commonInvariant_normalizedStateBlockForm_preservingBlockAction_jointSupport` then packages the normalized state blocks and the action of every preserving operation on that minimum joint support; ambient preserving operations inherit this action by restriction and intertwining. The finite invariant conditional family and one preserving Kraus realization are now constructed from SSA equality. The declaration `Matrix.exists_normalizedConditionalSliceBlockForm_preservingBlockAction_jointSupport` applies the block theorem to this family and retains the Petz middle-system channel together with the support intertwining and diagonal block action. The declaration `Matrix.exists_markovBipartiteBlockForm_jointSupport` uses the same support isometry and block coordinates to reconstruct the bipartite marginal. It produces positive unnormalized factors $\omega_j$, proves that their traces sum to one, and establishes $\bigoplus_j \sigma_j \otimes \omega_j$ on the minimum joint support, with the tensor order reversed from HJPW. This is a scope-restricted joint-support variant toward HJPW equation (14), not its ambient equivalence on $H_B$. The declaration `Matrix.exists_ambientMarkovBipartiteBlockForm` completes the support isometry to an ambient middle-system unitary and splits its orthogonal complement into one-dimensional zero-weight tensor sectors. It retains the supported positive factors, proves positivity on every ambient sector, and proves the total conditional trace is one. Thus HJPW equations (13)–(14), lines 493–502, now hold on the ambient $H_B$ without an extra source hypothesis. The declaration `Matrix.exists_markovDilationBlockForm` proves equation (15) for one chosen pure-ancilla unitary, together with supported Petz agreement, the exact recovery identity, and normalized sector output states on the supported common-factor sectors. The declaration `Matrix.MarkovDilationBlockForm.ambient_tripartite_block_form` then combines equations (11), (14), and (15). With $W = \mathbf{1}_A \otimes (U_B \otimes \mathbf{1}_C)$, it proves

$$W^\dagger \rho_{ABC} W = 0_\perp \oplus \bigoplus_j \omega_j \otimes \widehat{\rho}_j. \tag{20}$$

The theorem `Matrix.hayashi_ssa_equality_characterization_forward` normalizes these blocks into the quantum-Markov decomposition. Every supported right factor remains its sector output state, even when its weight is zero. A zero-weight supported sector may use a filler only on the left; a complementary sector may use fillers on both sides.

## 7 Classification

*Product-reference recovery and the structural implication are complete.* The general positive-semidefinite product reference, its marginal and support calculus, the global support-compressed raw formula, and the supported-input and positive-definite-left corollaries are formalized. The exact SSA specialization proves HJPW equation (11). The invariant-family decomposition, joint-support reduction, ambient equation (14), and recovery-dilation action of equation (15) give the ambient tripartite blocks. Their probability normalization and fibre reordering prove the Hayashi forward implication, so the SSA-equality characterization is axiom-free.

The raw/completed singular boundary is preserved. The generic completed singular product-reference channel is not asserted to factor as an identity on the first subsystem tensored with a local channel. The completed local channel is used only through its proved agreement with the raw support map on the supported marginal.

## References


[1] P. Hayden, R. Jozsa, D. Petz, and A. Winter, *Structure of states which satisfy strong subadditivity of quantum entropy with equality*, Communications in Mathematical Physics **246** (2004), 359–374; arXiv:quant-ph/0304007v2.