

The Unbroken Subgroup Is the Stabilizer of a Block

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Verdict: false-source (resolved).

Let a finite group G act by matrix product operators with three-cocycle ω , matrix product unitaries in the first source [GRS24], on the set $\mathcal{I}$ of blocks x of a symmetry-broken ground space, and assume, as both sources do, that the action on $\mathcal{I}$ is transitive [GRLM23, GRS24]. The two sources define the unbroken subgroup H differently, and only one of the definitions supports the classification of the solutions of the coupled pentagon equation for the L-symbols by H together with a point of a torsor for $\mathcal{H}^2(H, \mathbb{C}^\times)$ [GRLM23], which becomes a class $\alpha \in \mathcal{H}^2(H, \mathbb{C}^\times)$ once a trivialization of $\omega|_H$ is chosen. For finite H this group is isomorphic to $\mathcal{H}^2(H, U(1))$ [con26], the coefficient group of the first source [GRS24]. The classification of phases in the sources rests on this classification of L-symbols together with further hypotheses on the Hamiltonians and paths considered [GRLM23, GRS24], which are not discussed here.

1 The two definitions

Garre-Rubio and Schuch define H as the set of $h \in G$ with $h \cdot x = x$ for every $x \in \mathcal{I}$, and state that the ground-state degeneracy is $|G/H|$ [GRS24].¹ This is the kernel

$$K = \bigcap_{x \in \mathcal{I}} \text{Stab}(x) \tag{1}$$

of the action on blocks. Garre-Rubio, Lootens and Molnár, whose classification the first paper reviews, define H as the subgroup of elements that do not permute one chosen block x_0 , that is, the stabilizer $\text{Stab}(x_0)$, and state that the number of blocks is $|G/H|$ [GRLM23].²

For a transitive action $\text{Stab}(g \cdot x_0) = g \text{Stab}(x_0) g^{-1}$, so K is the normal core of $\text{Stab}(x_0)$, the largest normal subgroup of G contained in it. The two definitions

¹Source traceability: `Papers/2405.00439/MPU-DW.tex`, line 1880 and lines 2013–2036.

²Source traceability: `Papers/2203.12563/Resubmission.tex`, lines 734–761; the stabilizer is defined at line 736, the two-cocycle ψ_{x_0} at lines 748–749, and the reconstruction of the L-symbols from (H, ψ) at lines 755–760.

agree exactly when $\text{Stab}(x_0)$ is normal in G . This holds for every abelian G , and therefore for the cyclic examples worked out in the first source [GRS24]. It fails for members of the general construction of the second source, which builds a symmetry-broken state for every subgroup $H \leq G$ trivializing ω [GRLM23].³

2 Which definition the classification needs

The classification needs the stabilizer, for three reasons.

1. *Counting blocks.* By the orbit–stabilizer theorem $|\mathcal{I}| = |G/\text{Stab}(x_0)|$, while $|G/K| \geq |\mathcal{I}|$ with equality only when the stabilizer is normal. The degeneracy $|G/H|$ asserted in the first source [GRS24] holds for the stabilizer.
2. *Trivializing the cocycle.* The coupled pentagon equation of the second source [GRLM23],

$$L_{h,k}^x L_{g,hk}^x = \omega(g, h, k) L_{g,h}^{k \cdot x} L_{gh,k}^x, \quad (2)$$

restricted to g, h, k in a subgroup fixing the single block x , is the statement that ω restricted to that subgroup is a coboundary with potential L^x . Fixing one block suffices, so the argument gives the stronger conclusion for the larger subgroup $\text{Stab}(x_0) \supseteq K$.

3. *Reconstruction.* The two-cocycle ψ_{x_0} is defined on the stabilizer, and the L-symbols are rebuilt from (H, ψ) through a choice of coset representatives k_y with $y = k_y \cdot x_0$ and the unique factorization $gk_x = k_y h$ with $h \in H$. This factorization is the coset decomposition of G by $\text{Stab}(x_0)$; with K in its place the representatives do not correspond to blocks. The restriction of L^{x_0} to H is a trivializing two-cochain of $\omega|_H$; only after a choice of trivializing gauge β is $\psi_{x_0} = L^{x_0}/\beta$ a two-cocycle. For fixed (G, ω, H) the gauge classes of solutions therefore form a torsor for $\mathcal{H}^2(H, \mathbb{C}^\times)$, as the second source states [GRLM23], and the classifying data are the stabilizer H up to conjugacy together with a point of that torsor, the data classifying the indecomposable module categories of Vec_G^ω .

The domain-wall discussion of the first source also uses the stabilizer reading for normal H : when $\text{Stab}(x_0)$ is normal, every block has the same stabilizer, so $g \notin H$ implies $g \cdot x \neq x$ for every block x , as that source assumes for the interchange of domain walls [GRS24].

3 A non-normal example

Let $G = S_3$ act on $\mathcal{I} = \{1, 2, 3\}$ by permutations, realized by the on-site permutation representation of S_3 on $\mathbb{C}^3$ acting on the three product states $|x\rangle^{\otimes N}$;

³Source traceability: `Papers/2203.12563/REsubmission.tex`, lines 1997–2005 and 2098–2100.

the three-cocycle is trivial. The action is transitive, and

$$\text{Stab}(3) = \{e, (12)\}, \quad K = \{e\}. \quad (3)$$

The stabilizer has order two and is not normal, since $(13)(12)(13) = (23) \notin \text{Stab}(3)$. The stabilizer reading gives three blocks, $|S_3/\text{Stab}(3)| = 3$, which is correct; the kernel reading gives $|S_3/K| = 6$. The transposition $(12) \notin K$ fixes the block 3, so a group element outside the kernel may still act on some block as an unbroken symmetry.

4 The formal statement

The formal theorem that a subgroup fixing a block trivializes the restricted three-cocycle assumes a group of matrix product operators with normal tensors multiplying exactly as the group on every nonempty chain, normal block tensors of positive bond dimension, and that the operator of every g carries the periodic vector of every block x to that of $g \cdot x$ at every positive length. The source settings, an injective exact matrix product operator representation (a matrix product unitary one in the first source) permuting injective blocks [GRLM23, GRS24], satisfy these assumptions, and so does the example above, whose operator and block tensors have bond dimension one. Given these, the theorem takes an arbitrary homomorphism $f : H \rightarrow G$ and one block x with $f(a) \cdot x = x$ for all $a \in H$, that is, the stabilizer reading.⁴ It applies in particular to the kernel, which fixes every block.

Verdict: false as printed, resolved. With the unbroken subgroup defined as the kernel of the action, the printed degeneracy $|G/H|$ fails for S_3 acting transitively on three blocks. The correct statement, available in arXiv:2203.12563 [GRLM23], reads H as the stabilizer of one block; the kernel agrees with it whenever the stabilizer is normal, in particular for every abelian group, so the cyclic examples of arXiv:2405.00439 [GRS24] are unaffected.

References

- [con26] The TNLLean contributors. Circle and nonzero-complex coefficients in degree-two cohomology. Paper-gap note fbc25_circle_complex_units_cohomology, TNLLean, 2026. https://sirui-lu.com/TNLLean/paper-gaps/fbc25_circle_complex_units_cohomology.pdf.
- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.

⁴Formal declaration: `MPOTensor.GroupFamily.IsNormalRepresentation.isTrivialGaugeClass_comap_omega_of_fixed` in `TNLLean/MPS/Symmetry/MPOSymmetry/PermutedBlocks`.

- [GRS24] José Garre-Rubio and Norbert Schuch. Fractional domain wall statistics in spin chains with anomalous symmetries. arXiv:2405.00439v2, 2024.