

The Truncated-Symmetry Detector Depends on the Endpoints

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Verdict: false-source (open).

1 The printed claim

Let U be a matrix product unitary with $U^2 = \mathbf{1}$ exchanging two symmetry-broken ground states $|\psi_A\rangle$ and $|\psi_B\rangle$, and let ω be its anomaly. For an interval $[i, j]$, Ref. [GRS24] obtains a truncation $U^{[i, j]}$ by writing U , after blocking, as a unitary circuit of depth two with nearest-neighbour gates [CPGSV17] and dropping the gates outside the region. It asserts that $U^{[i, j]}$ is unitary, equals U inside the region, and creates a pair of domain walls around i and j . For sites $i_2 < i_1 < j_1 < j_2$ at mutual distances much larger than the correlation length, it then states

$$\langle \psi_A | (U^{[i_2, j_2]})^\dagger (U^{[i_1, j_1]})^\dagger U^{[i_2, j_2]} U^{[i_1, j_1]} | \psi_A \rangle = \omega, \quad (1)$$

and, for an on-site symmetry $U = u^{\otimes N}$ with the truncation $U^{[a, b]} = \bigotimes_{i \in [a, b]} u^{[i]}$, that the left side is 1.¹

2 A counterexample

Let X and Z be the Pauli matrices on $\mathbb{C}^2$, so that

$$X^2 = Z^2 = \mathbf{1}, \quad ZXZ = -X. \quad (2)$$

Take the on-site symmetry $U = X^{\otimes N}$ on a periodic chain of even length N . Its anomaly is trivial, $\omega = 1$, and $|\psi_A\rangle = |0 \cdots 0\rangle$, $|\psi_B\rangle = |1 \cdots 1\rangle$ are the two symmetry-broken ground states of $H = -\sum_i Z_i Z_{i+1}$.

¹Source traceability: `Papers/2405.00439/MPU-DW.tex`, lines 2077–2106; the display is `detecZ2`.

A depth-two circuit for U . Let the first layer apply the gate $X \otimes ZX$ to every pair of sites $(2k, 2k+1)$, and let the second layer apply the gate $Z \otimes \mathbf{1}$ to every pair $(2k+1, 2k+2)$. Since $Z \cdot ZX = X$, the product of the two layers is

$$\left(\prod_k Z_{2k+1}\right)\left(\prod_k X_{2k}(ZX)_{2k+1}\right) = X^{\otimes N}. \quad (3)$$

Let i be even and j odd. The gates contained in $[i, j]$ are the first-layer gates on all pairs $(2k, 2k+1)$ in $[i, j]$ and the second-layer gates on the pairs $(2k+1, 2k+2)$ with $2k+2 \leq j$. Every odd site of $[i, j]$ except j keeps both factors $Z \cdot ZX = X$, while the site j keeps only ZX . Dropping all other gates therefore gives

$$U^{[i,j]} = Z_j \prod_{k \in [i,j]} X_k. \quad (4)$$

It is unitary, equals U on $[i, j-1]$, and acts trivially outside $[i, j]$.

The detector. Let $i_2 < i_1 < j_1 < j_2$ with i_1, i_2 even and j_1, j_2 odd, and write $P_r = \prod_{k \in [i_r, j_r]} X_k$, so that $U^{[i_r, j_r]} = Z_{j_r} P_r$. The operators P_1 and P_2 commute and square to the identity. The site j_2 lies outside $[i_1, j_1]$, so Z_{j_2} commutes with P_1 and Z_{j_1} ; the site j_1 lies in $[i_2, j_2]$, so $Z_{j_1} P_2 Z_{j_1} = -P_2$ by (2). Hence

$$\begin{aligned} (U^{[i_2, j_2]})^\dagger (U^{[i_1, j_1]})^\dagger U^{[i_2, j_2]} U^{[i_1, j_1]} &= P_2 Z_{j_2} P_1 Z_{j_1} Z_{j_2} P_2 Z_{j_1} P_1 \\ &= P_2 P_1 (Z_{j_1} P_2 Z_{j_1}) P_1 \\ &= -\mathbf{1}. \end{aligned} \quad (5)$$

The expectation value on every normalized state, $|\psi_A\rangle$ included, is $-1 \neq \omega$. The truncations are circuit truncations of U in the sense of the source, the four sites can be arbitrarily far apart, and the data are nondegenerate, so (1) is false as printed. The plain truncation $X^{\otimes [a,b]}$ of the source's on-site remark comes from the circuit with first layer $X \otimes X$ and trivial second layer; the two circuits differ only by the unitary Z on the leg between the layers.

The general mechanism. The circuit of a matrix product unitary is determined only up to unitaries inserted between the layers, and the endpoint-coordinate truncation of Ref. [FRBC25] is determined only up to the identifications of its endpoint spaces [con26b]. Either freedom replaces $U^{[i_1, j_1]}$ by $WU^{[i_1, j_1]}$ for a unitary W near an endpoint. If W is near j_1 , where $U^{[i_2, j_2]}$ acts as U , then $U^{[i_2, j_2]}W = \widetilde{W}U^{[i_2, j_2]}$ with $\widetilde{W} = UWU^\dagger$, also near j_1 . The detector then becomes

$$\langle \psi_A | (U^{[i_2, j_2]})^\dagger (U^{[i_1, j_1]})^\dagger W^\dagger \widetilde{W} U^{[i_2, j_2]} U^{[i_1, j_1]} | \psi_A \rangle, \quad (6)$$

and the local operator $W^\dagger \widetilde{W}$ need not act by 1. In the counterexample, $W = Z_{j_1}$ and $W^\dagger \widetilde{W} = Z(XZX) = -\mathbf{1}$.

3 What the argument needs

The source’s argument treats $U^{[i,j]}$ on the states it meets as the domain-wall string $O^{[i,j]}$ of its Section III.E, whose endpoint tensors create the walls e_{AB} and e_{BA} on $|\psi_A\rangle$ and the walls e_{BA} and e_{AB} on $|\psi_B\rangle$, the second pair being the images of the first under U with the phases c_{AB} and c_{BA} . For a truncation the needed property is: *the walls that $U^{[i,j]}$ creates on $|\psi_B\rangle$ are the images under U of the walls it creates on $|\psi_A\rangle$* . With this property the two orders $U^{[i_2,j_2]}U^{[i_1,j_1]}|\psi_A\rangle$ and $U^{[i_1,j_1]}U^{[i_2,j_2]}|\psi_A\rangle$ are both multiples of one state with four domain walls, the first acquiring $c_{AB}c_{BA} = \omega$ when the outer truncation passes over the walls of the inner one, and (1) follows from unitarity. Without it the two states are built from different walls, and the detector compares them.

In the counterexample the property fails. The truncation $U^{[i,j]} = Z_j P$ sends $|\psi_A\rangle$ to $-|0 \cdots 01 \cdots 10 \cdots 0\rangle$ and $|\psi_B\rangle$ to $+|1 \cdots 10 \cdots 01 \cdots 1\rangle$, whereas U sends the first state to $-|1 \cdots 10 \cdots 01 \cdots 1\rangle$: the walls created on $|\psi_B\rangle$ differ from the images of those created on $|\psi_A\rangle$ by the sign of the wall at j .

4 The formal statements

The counterexample is formalized for any two sets of sites S_1, S_2 and sites $j_1 \in S_2, j_2 \notin S_1, j_1 \neq j_2$: the commutator of $Z_{j_1} X^{S_1}$ and $Z_{j_2} X^{S_2}$ is -1 .²

The detection is formalized as a scope restriction that makes the needed property a hypothesis. Let $O_1 = O^{[i_1,j_1]}$ and $O_2 = O^{[i_2,j_2]}$ be the domain-wall strings of Ref. [GRS24] with the three inner regions longer than the buffer of the local action [con26a]. If U_1 and U_2 are unitary with

$$U_2 U_1 |\psi_A\rangle = O_2 O_1 |\psi_A\rangle, \quad U_1 U_2 |\psi_A\rangle = O_1 O_2 |\psi_A\rangle, \quad (7)$$

then

$$\langle \psi_A | U_2^\dagger U_1^\dagger U_2 U_1 | \psi_A \rangle = c_{AB} c_{BA} \langle \psi_A | \psi_A \rangle = \omega(g, g, g) \omega(g, e, g) \langle \psi_A | \psi_A \rangle, \quad (8)$$

where the second equality holds for an involution g of a normal representation, and $\omega(g, e, g) = 1$ for fusion tensors trivial on the identity, as in the source.³ The hypotheses (7) hold for truncations with the property of the previous section, and fail in the counterexample.

Verdict: false as printed, open. The value of (1) depends on the choice of truncation; it equals ω for truncations whose walls on $|\psi_B\rangle$ are the images under

²Formal declarations: `Matrix.groupCommutator_dressed_onSiteTruncation_pauliX` and `Matrix.star_dotProduct_groupCommutator_dressed_onSiteTruncation_pauliX` in `TNLean/MPS/MPU/TruncationCommutator`.

³Formal declarations: `MPOTensor.GroupFamily.BlockActionData.star_dotProduct_groupCommutator_mulVec_mpv_of_wallString` and `MPOTensor.GroupFamily.BlockActionData.star_dotProduct_groupCommutator_mulVec_mpv_eq_omega` in `TNLean/MPS/Symmetry/MPOSymmetry/DomainWallDetector`.

U of their walls on $|\psi_A\rangle$. Elimination plan: identify a class of circuit or endpoint-coordinate truncations with this property and prove (7) for it. The same dependence must be examined for the detection of the general-group phases **detectG** (lines 2108–2114).

References

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- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.
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- [GRS24] José Garre-Rubio and Norbert Schuch. Fractional domain wall statistics in spin chains with anomalous symmetries. arXiv:2405.00439v2, 2024.