

Exchange of Domain-Wall Endpoints on Glued Half-Chains

2026-10-01

Verdict: scope-restriction (open).

Let U be a matrix product unitary of order two exchanging the injective ground states $|\psi_A\rangle$ and $|\psi_B\rangle$, let e_{AB} and e_{BA} be domain walls on which U acts with the phases c_{AB} and c_{BA} , and let $O^{[i,j]}$ be the string of U on the sites $i, \dots, j$ whose two open virtual legs are closed by the endpoint tensors built from left inverses of the ground-state tensors [GRS24]. Ref. [GRS24] cuts such a string in the middle and obtains the half-chain operator $O_x^{[i]}$: the left endpoint at i followed by the tensors of U , with the bond of the operator string fixed to the value x at a cut on the right. It proves, for $i > j$,

$$O_x^{[j]} O_y^{[i]} |\psi_A\rangle = c_{AB} O_x^{[i]} O_y^{[j]} |\psi_A\rangle, \quad (1)$$

as an identity between half-chain objects with the two open legs x and y , and remarks that such an object must be glued with the other half of a string to give a state.¹

1 The formal statement

The formalization glues both sides of (1) with right endpoints at two sites $r_1 \neq r_2$ to the right of i , the leg x being closed at r_1 and the leg y at r_2 . The relation becomes an identity of periodic states,

$$O^{[j,r_1]} O^{[i,r_2]} |\psi_A\rangle = c_{AB} O^{[i,r_1]} O^{[j,r_2]} |\psi_A\rangle, \quad (2)$$

and it is proved for both orders of r_1 and r_2 , whenever the regions between the four endpoints are longer than the buffer of the local action.² For $r_2 < r_1$ the left side is $c_{AB}c_{BA}$ times the periodic state with the domain walls $e_{AB}, e_{BA}, e_{AB}, e_{BA}$ at j, i, r_2, r_1 , and the right side is c_{BA} times the same state; for $r_1 < r_2$ the two sides are c_{AB} and 1 times the state with these walls at j, i, r_1, r_2 .

¹Source traceability: `Papers/2405.00439/MPU-DW.tex`, lines 1427–1659; the remark on gluing is at line 1489.

²Formal declarations: `MPOTensor.GroupFamily.BlockActionData.wallString_mul_wallString_mulVec_mpv_swap_left_far`, `MPOTensor.GroupFamily.BlockActionData.wallString_mul_wallString_mulVec_mpv_swap_left_near`. The buffer is recorded in Ref. [con26].

2 Comparison

Equation (2) follows from (1) by contracting both sides with the same right halves, so it is a consequence of the source statement; the converse does not follow, since only right halves of domain-wall strings are contracted. The open-leg identity (1) uses in addition the fixed-point form of the action tensors, which writes $m + 1$ sites of U acting on A as one acted site, the right action tensor, $m - 1$ sites of B , the left action tensor and one acted site. The formal action tensors satisfy only the reduction of m acted sites to B^m between the two action tensors, so the half-chain objects are not available with open legs.

Verdict: open scope restriction. Equation (2) is the formalized content of the single-endpoint exchange. The open-leg identity (1) needs the fixed-point decomposition of the acted chain as an additional property of the action tensors.

References

- [con26] The TNLLean contributors. Domain walls: Nonzero tensors, phases and blocked local actions. Paper-gap note gs24_domain_wall_nondegenerate, TNLLean, 2026. https://sirui-lu.com/TNLLean/paper-gaps/gs24_domain_wall_nondegenerate.pdf.
- [GRS24] José Garre-Rubio and Norbert Schuch. Fractional domain wall statistics in spin chains with anomalous symmetries. arXiv:2405.00439v2, 2024.