

The Left Action Vectors of the CZX Product States

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Verdict: local-correction (resolved).

The second computation of the CZX anomaly in Ref. [GRS24] obtains $\omega = -1$ from the action tensors of the decorated CZX matrix product unitary U on the product states $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$, which U exchanges, through the ratio $L_0/L_1 = -1$.¹ The conclusion is correct, but the printed left action vectors are not.

1 The tensor and the action

The tensor of U , with physical output i , input j , left bond l and right bond r , is

$$T_{lr}^{ij} = \delta_{i,1-j} \hat{H}_{l,1-j} \delta_{r,1-j} (-1)^{1-j}, \quad \hat{H} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (1)$$

The reading (1) reproduces the printed fusion tensors $\hat{V} = \langle 1| \otimes \langle 0|$ and $V = -|\hat{-}\rangle \otimes |\hat{+}\rangle$, with $|\hat{\pm}\rangle = |0\rangle \pm |1\rangle$, and the sign -1 of their associator.² The action of T on the product state $|s\rangle$ has the single nonzero letter

$$B_0^1 = T^{10} = -|\hat{-}\rangle \langle 1|, \quad B_1^0 = T^{01} = |\hat{+}\rangle \langle 0|. \quad (2)$$

2 The printed vectors

The source draws, for three sites, the decomposition $B^a B^b B^c = B^a w \delta_{b,1-s} v B^c$ of the action on $|s\rangle^{\otimes 3}$ by the letters (2), with right vector w after the first site and left vector v before the third. It prints $w = |\hat{+}\rangle$, $v = \langle \hat{+}|$ for $s = 0$, and $w = -|\hat{+}\rangle$, $v = \langle \hat{+}|$ for $s = 1$. For $s = 0$ these give $vw = 2$ and $vB_0^1 w = \langle \hat{+}|\hat{-}\rangle \cdot (\dots) = 0$, so the right-hand side of the drawn decomposition vanishes although $B_0^1 B_0^1 B_0^1 \neq 0$. For $s = 1$ they give $vw = -2$. No choice among the four

¹Source traceability: `Papers/2405.00439/MPU-DW.tex`, lines 1246–1339; the printed vectors are at lines 1272–1273 and 1300–1301.

²Formal declarations: `CZXCompression.czxDoratedTensor`, `CZXCompression.czxDorated_associator_eq_neg_one`.

readings of the diagram (the Hadamard on the left or right bond, the physical input at the bottom or top), and no normalization factor, makes $v = \langle \hat{\tau} |$ work.

3 The correction

Keeping the printed right vectors, the conditions $vw = 1$, $vB_s^{1-s}w = 1$ and $B_s^{1-s}wvB_s^{1-s} = (B_s^{1-s})^2$ force

$$v = \langle 1 | \quad (s = 0), \quad v \in \{(c, d) : c + d = -1\}, \text{ for instance } -\langle 0 | \quad (s = 1). \quad (3)$$

With the choices (3), $v = \langle 1 |$ and $v = -\langle 0 |$, both three-site decompositions hold and (v, w) are reductions of the action onto the exchanged state. The resulting L-symbols are $L_{s,s}^0 = 1$ and $L_{s,s}^1 = -1$; the source's values $L_i = (-1)^{i+1}$ differ from these by an action-tensor gauge, and the gauge-invariant ratio $L_0/L_1 = -1 = \omega(s, s, s)$ of line 1335 is unchanged.³

Verdict: resolved local fix. The left action vectors at lines 1272 and 1300 should read $\langle 1 |$ and $-\langle 0 |$; the anomaly computed from them is the printed $\omega = -1$.

References

- [GRS24] José Garre-Rubio and Norbert Schuch. Fractional domain wall statistics in spin chains with anomalous symmetries. arXiv:2405.00439v2, 2024.

³Formal declarations: `CZXCompression.czxBLOCKActionData`, `CZXCompression.czX_omega_eq_lSymbol_ratio`, `CZXCompression.czX_lSymbol_ratio_eq_neg_one` in `TNLean/MPS/Symmetry/MP0Symmetry/CZXPermutedBlocks`.