

Star-Closure Correction for the GNVW Support-Algebra Lemma

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Verdict: local-correction (resolved).

1 The printed statement

Section 7.1, Lemma 7 of arXiv:0910.3675v2 assumes that $\mathcal{A} \subseteq \mathcal{B}_1 \otimes \mathcal{B}_2$ is a subalgebra [GNVW12]. It defines $\text{Spp}(\mathcal{A}, \mathcal{B}_1)$ as the smallest C^* -subalgebra generated by the first-factor coefficients and asserts

$$\text{Spp}(\mathcal{A}, \mathcal{B}_1)' = \{b \in \mathcal{B}_1 : b \otimes \mathbb{1} \in \mathcal{A}'\}. \quad (1)$$

The statement and leastness argument occupy lines 1181–1191 of the v2 source, coefficient generation appears at lines 1188–1191, and the commutant argument occupies lines 1192–1208. The source label is `lem:spp`. The published article has DOI [10.1007/s00220-012-1423-1](https://doi.org/10.1007/s00220-012-1423-1).

Coefficient generation and leastness remain valid for an arbitrary algebra input. Coefficient expansion proves containment, and dual-basis coefficient maps prove leastness. The commutant identity (1), however, requires closure of the input algebra under the involution.

2 Counterexample and correction

Take $\mathcal{B}_1 = M_2(\mathbb{C})$ and $\mathcal{B}_2 = \mathbb{C}$, and define

$$\mathcal{A} = \text{span}\{\mathbb{1}, E_{12}\} \subset M_2(\mathbb{C}). \quad (2)$$

The algebra $\mathcal{A}$ is unital but not closed under adjoints. Its coefficient set generates $M_2(\mathbb{C})$ as a C^* -algebra because it contains E_{12} and therefore also generates $E_{21} = E_{12}^*$. Hence

$$\text{Spp}(\mathcal{A}, \mathcal{B}_1)' = \mathbb{C}\mathbb{1}. \quad (3)$$

By contrast, direct computation gives

$$\mathcal{A}' = \text{span}\{1, E_{12}\}. \quad (4)$$

Equations (3) and (4) contradict the printed identity (1). Thus the commutant clause is false for a general non-star-closed algebra.

To state coefficient generation, leastness, and the commutant clause as one package, assume that $\mathcal{A}$ is a C^* -subalgebra. In the finite-dimensional formal setting, this is equivalently a star-subalgebra. For an element of $\mathcal{A}$, the adjoint of each coefficient is an appropriately indexed coefficient of an adjoint element of $\mathcal{A}$. This supplies the involution closure used in the commutant argument. The correction does not claim that coefficient generation or leastness alone requires star closure.

3 Star closure in the applications

Schumacher and Werner distinguish the algebraic support construction from the closure property used for observable algebras [SW04]. At lines 1161–1165 of arXiv:quant-ph/0405174, they define the support algebra as the ordinary algebra generated by the coefficient support. At lines 1170–1171, they deduce adjoint closure from adjoint closure of the input observable algebra. This is a conditional closure argument; it does not identify every “subalgebra” with a “ C^* -subalgebra.”

The GNVW quantum-cellular-automaton application satisfies the corrected hypothesis. At lines 1251–1265 of arXiv:0910.3675v2, the input is the image of a local observable algebra under a $*$ -automorphism [GNVW12]. The matrix-product-unitary application has the same form. At lines 2300–2306 of arXiv:1703.09188, the map ω preserves the local star-algebra operations [CPGSV17]; at lines 2313–2334, the support algebra is formed from

$$\omega(\mathcal{A}_{2x} \otimes \mathcal{A}_{2x+1}). \quad (5)$$

This input is the image of a local full matrix star-algebra under a star-automorphism and is therefore star closed.

4 Formal resolution and remaining scope

The formal support-algebra API takes a star-subalgebra as input. This is the corrected hypothesis, and both intended QCA and MPU applications provide it. The local correction is therefore resolved; the false commutant characterization for non-star-closed inputs need not be formalized.

The distinct restriction to full complex matrix factors remains open and is recorded in `docs/paper-gaps/gnvw12_support_algebra_full_matrix_scope.tex`. Removing that restriction requires a support-algebra construction for arbitrary finite-dimensional C^* -algebra factors. It does not require weakening the corrected star-closed input.

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.
- [GNVW12] David Gross, Vincent Nesme, Héctor Vogts, and Reinhard F. Werner. Index theory of one dimensional quantum walks and cellular automata. *Communications in Mathematical Physics*, 310(2):419–454, 2012.
- [SW04] Benjamin Schumacher and Reinhard F. Werner. Reversible quantum cellular automata. Preprint, 2004.