

Outer-Factor Basis Correction in the Support-Algebra Commutation Lemma

2026-08-24

Verdict: local-correction (resolved).

1 The printed coefficient argument

Let $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ be finite-dimensional algebras, and let

$$\mathcal{A}_1 \subseteq \mathcal{B}_1 \otimes \mathcal{B}_2, \tag{1}$$

$$\mathcal{A}_2 \subseteq \mathcal{B}_2 \otimes \mathcal{B}_3. \tag{2}$$

The support-algebra commutation lemma assumes that $\mathcal{A}_1 \otimes \text{Id}_3$ commutes with $\text{Id}_1 \otimes \mathcal{A}_2$. Its coefficient proof chooses bases and, for $a \in \mathcal{A}_1$ and $a' \in \mathcal{A}_2$, writes

$$a = \sum_{\mu} e_{\mu} \otimes a_{\mu}, \tag{3}$$

$$a' = \sum_{\nu} a'_{\nu} \otimes e'_{\nu}. \tag{4}$$

Commutation of the lifted elements then gives

$$0 = [a \otimes \text{Id}_3, \text{Id}_1 \otimes a'] \tag{5}$$

$$= \sum_{\mu, \nu} e_{\mu} \otimes [a_{\mu}, a'_{\nu}] \otimes e'_{\nu}. \tag{6}$$

This argument appears in the Schumacher–Werner lemma labelled **sppcomm**, at lines 1174–1194 of arXiv:quant-ph/0405174 [SW04], and in the GNVW lemma with the same source label, at lines 1221–1246 of arXiv:0910.3675v2 [GNVW12]. The published GNVW article has DOI [10.1007/s00220-012-1423-1](https://doi.org/10.1007/s00220-012-1423-1).

2 The outer-factor correction

Both printed proofs place the basis elements e'_ν in $\mathcal{B}_2$. That assignment is incompatible with (4). Because $a' \in \mathcal{B}_2 \otimes \mathcal{B}_3$, its coefficients a'_ν lie in $\mathcal{B}_2$, whereas its basis elements e'_ν lie in $\mathcal{B}_3$. The printed assignment is also incompatible with the next proof step, which treats the tensors $e_\mu \otimes e'_\nu$ as a basis of $\mathcal{B}_1 \otimes \mathcal{B}_3$.

The corrected types are

$$e_\mu \in \mathcal{B}_1, \tag{7}$$

$$a_\mu, a'_\nu \in \mathcal{B}_2, \tag{8}$$

$$e'_\nu \in \mathcal{B}_3. \tag{9}$$

With these assignments, the outer tensors in (6) are linearly independent. Therefore

$$[a_\mu, a'_\nu] = 0 \quad \text{for every } \mu \text{ and } \nu. \tag{10}$$

The correction changes neither the hypotheses nor the conclusion of the lemma. It restores the factor type required by the coefficient argument.

3 Formal resolution

The formal coefficient statement indexes the blocks of a by the first factor and the slices of a' by the third factor. Its conclusion concerns matrices on the common second factor, so it directly implements the corrected types above. Pairwise commutation of the coefficients is formalized by `Matrix.middleSlices_commute_of_overlappingLifts_commute`; passage to the generated support algebras is formalized by `Matrix.supportAlgebras_commute_of_overlappingLifts_commute`.

The separate star-closure correction is documented in `docs/paper-gaps/gnvw12_support_algebra_star_closure.tex`. The distinct restriction to full complex matrix factors remains open and is documented in `docs/paper-gaps/gnvw12_support_algebra_full_matrix_scope.tex`.

4 Verdict

Replacing $\mathcal{B}_2$ by $\mathcal{B}_3$ as the ambient algebra of the basis $\{e'_\nu\}_\nu$ is a resolved local correction. Both the typed expansion of a' and the outer-factor linear-independence argument require this replacement.

References

- [GNVW12] David Gross, Vincent Nesme, Héctor Vogts, and Reinhard F. Werner. Index theory of one dimensional quantum walks and cellular automata. *Communications in Mathematical Physics*, 310(2):419–454, 2012.
- [SW04] Benjamin Schumacher and Reinhard F. Werner. Reversible quantum cellular automata. Preprint, 2004.