

Full-Matrix Scope for the GNVW Support-Algebra Formalization

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Verdict: scope-restriction (open).

1 The general support-algebra statement

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be finite-dimensional C^* -algebras, and let $\mathcal{A} \subseteq \mathcal{B}_1 \otimes \mathcal{B}_2$ be a C^* -subalgebra. There is a smallest C^* -subalgebra $\mathcal{C}_1 \subseteq \mathcal{B}_1$ such that

$$\mathcal{A} \subseteq \mathcal{C}_1 \otimes \mathcal{B}_2, \quad (1)$$

$$\mathcal{C}_1 = \text{Spp}(\mathcal{A}, \mathcal{B}_1). \quad (2)$$

Choose a basis $(e_\mu)_\mu$ of $\mathcal{B}_2$. Every $a \in \mathcal{A}$ has a unique expansion

$$a = \sum_{\mu} a_{\mu} \otimes e_{\mu}. \quad (3)$$

The support algebra $\text{Spp}(\mathcal{A}, \mathcal{B}_1)$ is generated by all coefficients a_{μ} obtained from all $a \in \mathcal{A}$. Its commutant inside $\mathcal{B}_1$ is

$$\text{Spp}(\mathcal{A}, \mathcal{B}_1)' = \{b \in \mathcal{B}_1 : b \otimes \mathbf{1} \in \mathcal{A}'\}. \quad (4)$$

These are the three assertions of Lemma 7 in Section 7.1 of Ref. [\[GNVW12\]](#), after applying the necessary star-closure correction to the commutant clause. That correction is recorded separately in `docs/paper-gaps/gnvw12_support_algebra_star_closure.tex`. In the arXiv v2 source, the statement and leastness argument appear at lines 1181–1191, the coefficient-generation argument appears at lines 1188–1191, and the commutant assertion appears at lines 1192–1208. The published DOI [10.1007/s00220-012-1423-1](#) and the arXiv v2 source are third-party-verifiable anchors. The source label for the lemma is `lem:spp`.

Applying the construction to both factors gives

$$\mathcal{A} \subseteq \text{Spp}(\mathcal{A}, \mathcal{B}_1) \otimes \text{Spp}(\mathcal{A}, \mathcal{B}_2), \quad (5)$$

$$\text{Spp}(\mathcal{A}, \mathcal{B}_1) \otimes \text{Spp}(\mathcal{A}, \mathcal{B}_2) \subseteq \mathcal{B}_1 \otimes \mathcal{B}_2. \quad (6)$$

The first inclusion is GNVW equation (22), which follows after Lemma 7 and is not one of that lemma's three clauses [GNVW12]. It appears at lines 1211–1219 of the arXiv v2 source with label `a2Supp`. Schumacher and Werner give the preceding coefficient-space and generated-support-algebra constructions in Section 4.3 of Ref. [SW04]; see lines 1137–1169 and the source labels `a2supp` and `a2Supp`.

2 The full-matrix formal statement

Let I and J be finite sets, and let

$$\mathcal{A} \subseteq M_{I \times J}(\mathbb{C}), \quad (7)$$

$$M_{I \times J}(\mathbb{C}) \cong M_I(\mathbb{C}) \otimes M_J(\mathbb{C}). \quad (8)$$

Assume that $\mathcal{A}$ is a star-subalgebra. This assumption implements the resolved local correction described above. The remaining open scope concerns only the extension from full complex matrix factors to arbitrary finite-dimensional C^* -algebra factors.

For $X \in \mathcal{A}$, define its left and right coefficient matrices by

$$X_{rs}^L(i, j) = X((i, r), (j, s)), \quad (9)$$

$$X_{ij}^R(r, s) = X((i, r), (j, s)). \quad (10)$$

The formal support algebras are the generated star-algebras

$$\text{Spp}_L(\mathcal{A}) = \text{alg}^* \{ X_{rs}^L : X \in \mathcal{A}, r, s \in J \}, \quad (11)$$

$$\text{Spp}_R(\mathcal{A}) = \text{alg}^* \{ X_{ij}^R : X \in \mathcal{A}, i, j \in I \}. \quad (12)$$

For a star-subalgebra $\mathcal{D} \subseteq M_I(\mathbb{C})$, left-support leastness is equivalent to coefficient containment:

$$\text{Spp}_L(\mathcal{A}) \subseteq \mathcal{D} \iff X_{rs}^L \in \mathcal{D} \text{ for all } X \in \mathcal{A}, r, s \in J. \quad (13)$$

If

$$I = \{0, \dots, p-1\}, \quad (14)$$

$$J = \{0, \dots, q-1\}, \quad (15)$$

then coefficient leastness is equivalent to tensor-factor leastness:

$$\text{Spp}_L(\mathcal{A}) \subseteq \mathcal{D} \iff \mathcal{A} \subseteq \mathcal{D} \otimes M_q(\mathbb{C}). \quad (16)$$

The commutant theorem states

$$B \otimes \mathbf{1}_J \in \mathcal{A}' \iff B \in \text{Spp}_L(\mathcal{A})'. \quad (17)$$

The right-factor statements are analogous.

For subspaces $S \subseteq M_I(\mathbb{C})$ and $T \subseteq M_J(\mathbb{C})$, define

$$S \boxtimes T := \text{span}_{\mathbb{C}} \{A \otimes B : A \in S, B \in T\}. \quad (18)$$

The formal counterpart of GNVW equation (22) of Ref. [GNVW12] is the literal containment

$$\mathcal{A} \subseteq \text{Spp}_L(\mathcal{A}) \boxtimes \text{Spp}_R(\mathcal{A}). \quad (19)$$

This statement places every element of $\mathcal{A}$ in the span of simple tensors whose factors lie in the two support algebras. It is stronger in form than merely asserting both one-sided containments.

The formal definitions and theorems are `Matrix.leftSupportAlgebra`, `Matrix.rightSupportAlgebra`, `Matrix.left_support_algebra_le_iff`, `Matrix.left_support_algebra_le_iff_le_kronecker_submodule`, `Matrix.le_support_kroneckerSubmodule`, `Matrix.left_kronecker_embed_mem_centralizer_iff`, and their right-factor counterparts in [TNLean/QCA/BipartiteSupportAlgebra](#).

3 Why the specialization suffices for the MPU application

The matrix-product-unitary Appendix observes that every one-site algebra is isomorphic to a full matrix algebra,

$$\mathcal{A}_z \cong M_d(\mathbb{C}), \quad (20)$$

before invoking the GNVW support algebra; see lines 2320–2334 of Ref. [CPGSV17]. Each two-site factor is therefore isomorphic to

$$\mathcal{A}_z \otimes \mathcal{A}_{z+1} \cong M_{d^2}(\mathbb{C}). \quad (21)$$

After choosing these matrix coordinates, the support algebra $\mathcal{R}_{2x}$ is an instance of (13) and (16). Consequently, the full-matrix theorem suffices for the support-algebra step in the spin-chain argument of Ref. [CPGSV17].

The star-closed-input correction and its verification for the GNVW and MPU applications are recorded in `docs/paper-gaps/gnvw12_support_algebra_star_closure.tex`. They are not part of the open scope considered here.

The restriction to full matrix factors is an open formalization boundary in TNLLean. It is not a mathematical gap in Lemma 7 or equation (22) of Ref. [GNVW12].

4 Elimination plan

Removing the full-matrix restriction requires the same construction for arbitrary finite-dimensional C^* -algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ and a C^* -subalgebra $\mathcal{A} \subseteq \mathcal{B}_1 \otimes \mathcal{B}_2$. Choose a basis $(e_\mu)_\mu$ of $\mathcal{B}_2$ and its dual basis $(\omega_\mu)_\mu$. Extract the coefficients by

$$a_\mu = (\text{id} \otimes \omega_\mu)(a). \quad (22)$$

Define $\text{Spp}(\mathcal{A}, \mathcal{B}_1)$ as the C^* -subalgebra generated by these coefficients.

The dual-basis expansion (3) proves the leastness property (1). Expanding the commutator gives

$$[b \otimes \mathbf{1}, a] = \sum_{\mu} [b, a_\mu] \otimes e_\mu. \quad (23)$$

This identity proves (4). Repeating the construction for the second factor proves (5). The existing full-matrix statements should then follow by taking matrix units as the chosen basis and using the associated coefficient functionals. This is the direct coefficient argument of Ref. [GNVW12]; it requires neither a bicommutant theorem nor a density theorem.

Until this generic construction is formalized, a source-labelled entry may carry a completion marker only if it states the full complex matrix-factor restriction explicitly.

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.

- [GNVW12] David Gross, Vincent Nesme, Héctor Vogts, and Reinhard F. Werner. Index theory of one dimensional quantum walks and cellular automata. *Communications in Mathematical Physics*, 310(2):419–454, 2012.
- [SW04] Benjamin Schumacher and Reinhard F. Werner. Reversible quantum cellular automata. Preprint, 2004.