

# Homogeneous-Chain Scope for the GNVW Support Algebras, Adjacent Generation, and Index

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**Verdict:** scope-restriction (open).

## 1 The site-dependent GNVW construction

Gross–Nesme–Vogts–Werner allow the cell algebra at each site  $y \in \mathbb{Z}$  to have a site-dependent positive matrix size [GNVW12, Section 7.2]:

$$\mathcal{A}_y = M_{d(y)}(\mathbb{C}). \quad (1)$$

For each  $x \in \mathbb{Z}$ , define the even pair and its left and right neighbouring pairs by

$$E_x = \{2x, 2x + 1\}, \quad (2)$$

$$L_x = \{2x - 1, 2x\}, \quad (3)$$

$$P_x = \{2x + 1, 2x + 2\}. \quad (4)$$

For a nearest-neighbour automorphism  $\alpha$ , locality gives

$$\alpha(\mathcal{A}_{E_x}) \subseteq \mathcal{A}_{L_x} \otimes \mathcal{A}_{P_x}. \quad (5)$$

Using the factor order in (5), source equation **RR2x** defines the support algebras

$$\mathcal{R}_{2x} = \text{Spp}_L(\alpha(\mathcal{A}_{E_x})) \subseteq \mathcal{A}_{L_x}, \quad (6)$$

$$\mathcal{R}_{2x+1} = \text{Spp}_R(\alpha(\mathcal{A}_{E_x})) \subseteq \mathcal{A}_{P_x}. \quad (7)$$

These definitions appear at lines 1251–1266 of the arXiv v2 source for Ref. [GNVW12]. The site-dependent cell structure and the causal-automorphism assumptions appear at lines 557–600. Neither (5) nor (6)–(7) requires translation covariance.

## 2 Adjacent generation and dimensions

Let  $r(y) > 0$  denote the matrix size of the support algebra at site  $y$ :

$$\mathcal{R}_y \cong M_{r(y)}(\mathbb{C}). \quad (8)$$

The adjacent-generation argument yields two ordered product identities:

$$d(2x)d(2x+1) = r(2x)r(2x+1), \quad (9)$$

$$r(2x+1)r(2x+2) = d(2x+1)d(2x+2). \quad (10)$$

The first identity is the matrix-size consequence of source equation **AARReven**, lines 1276–1287, and the second follows from source equation **AARRodd**, lines 1306–1315, of Ref. [GNVW12]. Dividing these identities gives

$$\frac{r(2x)}{d(2x)} = \frac{d(2x+1)}{r(2x+1)}, \quad (11)$$

$$\frac{d(2x+1)}{r(2x+1)} = \frac{r(2x+2)}{d(2x+2)}. \quad (12)$$

Together, these equalities are the site-independent multiplicative index relation:

$$\frac{r(2x)}{d(2x)} = \frac{d(2x+1)}{r(2x+1)} = \frac{r(2x+2)}{d(2x+2)}. \quad (13)$$

The source labels this relation **dimtransfer**; it appears at lines 1317–1329 of Ref. [GNVW12].

## 3 The homogeneous specialization

The present spin-chain algebra uses one fixed positive on-site matrix size  $d$ . Hence

$$d(y) = d \quad \text{for every } y \in \mathbb{Z}. \quad (14)$$

Under (14), the dimension identities become

$$d^2 = r(2x)r(2x+1), \quad (15)$$

$$r(2x+1)r(2x+2) = d^2, \quad (16)$$

$$\frac{r(2x)}{d} = \frac{d}{r(2x+1)} = \frac{r(2x+2)}{d}. \quad (17)$$

Cirac–Pérez-García–Schuch–Verstraete use this homogeneous setting: their Appendix fixes  $\mathcal{A}_y \cong M_d(\mathbb{C})$  and constructs the even support algebra at lines 2320–2329 [CPGSV17].

The theorems currently proved for the fixed- $d$  spin chain establish the homogeneous forms of (5) and (6)–(7), together with two-sided support containment. They also prove that the canonically embedded support algebras at distinct site indices commute pairwise. This is the homogeneous specialization of the consecutive/disjoint commutation argument at lines 1270–1274 of Ref. [GNVW12]; it does not use translation covariance.<sup>1</sup> The fixed- $d$  development does not classify the support algebras as full matrix factors, prove the adjacent-generation equalities or their dimension consequences (15)–(17), or construct the GNVW index. Extending the construction and the commutation theorem to site-dependent cell sizes also remains outside the present scope.

This restriction differs from the full-matrix-factor restriction recorded in `docs/paper-gaps/gnvw12_support_algebra_full_matrix_scope.tex`. That note compares full complex matrix factors with arbitrary finite-dimensional  $C^*$ -algebras. The present note compares a fixed matrix size along the chain with the site-dependent full matrix sizes allowed by Ref. [GNVW12].

## 4 Elimination plan

Removing this restriction requires an algebraic local-observable system with site-dependent cell algebras (1), canonical inclusions for arbitrary finite regions, and a norm completion. The finite-region restriction of a causal automorphism must then be expressed in the ordered  $L_x \times P_x$  coordinates of (5).

The same coefficient definitions yield the support algebras (6) and (7). Support commutation, the scalar-center argument, and adjacent generation then give (9) and (10) without assuming that  $d(y)$  is constant. The ratio (13) then defines the site-independent GNVW index. The fixed- $d$  theorems should follow by specializing  $d(y)$  to the constant value  $d$ .

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<sup>1</sup>Formal declarations: `SpinChain.PropagatesWithin.evenPairLocalImage`, `SpinChain.PropagatesWithin.evenSupportAlgebra`, `SpinChain.PropagatesWithin.oddSupportAlgebra`, and `SpinChain.PropagatesWithin.evenPairLocalImage_le_support_kroneckerSubmodule` in [TNLean/QCA/SiteIndexedSupportAlgebra](#), and `SpinChain.PropagatesWithin.embeddedSupportAlgebra_pairwise_commute` in [TNLean/QCA/SupportAlgebraCommutation](#).

Until this site-dependent observable algebra is available, the corresponding formal statements remain restricted to homogeneous chains.

## References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.
- [GNVW12] David Gross, Vincent Nesme, Héctor Vogts, and Reinhard F. Werner. Index theory of one dimensional quantum walks and cellular automata. *Communications in Mathematical Physics*, 310(2):419–454, 2012.