

Single-Block Scope for the Schumacher–Werner Commuting-Algebra Proposition

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Verdict: scope-restriction (open).

1 The source proposition

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are commuting finite-dimensional subalgebras of $\mathcal{B}(\mathcal{H})$ with decompositions

$$\mathcal{A} \cong \bigoplus_{\mu} M_{n(\mu)}(\mathbb{C}), \quad (1)$$

$$\mathcal{B} \cong \bigoplus_{\nu} M_{m(\nu)}(\mathbb{C}). \quad (2)$$

Proposition **Cscom** of Schumacher and Werner states that the product algebra $\mathcal{AB}$ decomposes into the block-pair products. The representation on $\mathcal{H}$ assigns a nonnegative integer multiplicity $r_{\mu\nu}$ to each pair and satisfies

$$\sum_{\mu, \nu} r_{\mu\nu} n(\mu) m(\nu) = \dim \mathcal{H}. \quad (3)$$

This is Proposition **Cscom** and Equation **multip** at lines 2119–2138 of Ref. [SW04]. For a fixed pair of blocks, multiplication gives a representation

$$M_{n(\mu)}(\mathbb{C}) \otimes M_{m(\nu)}(\mathbb{C}) \longrightarrow \mathcal{B}(\mathcal{H}), \quad (4)$$

$$A_{\mu} \otimes B_{\nu} \longmapsto A_{\mu} B_{\nu}. \quad (5)$$

The source allows this block-pair representation to vanish because a block identity need not equal the identity on $\mathcal{H}$ [SW04].

2 The formal specialization

The formal result takes one unital block in each factor. Let I be a finite nonempty set, let $\mathcal{A}, \mathcal{B} \subseteq M_I(\mathbb{C})$ be unital star-subalgebras, and assume

$$\mathcal{A} \cong M_p(\mathbb{C}), \quad (6)$$

$$\mathcal{B} \cong M_q(\mathbb{C}), \quad (7)$$

where $p, q > 0$. Since the two subalgebras share the ambient identity, their multiplication representation is unital and therefore nonzero. Simplicity of $M_{pq}(\mathbb{C})$ makes it injective. Consequently,

$$\mathcal{A} \vee \mathcal{B} \cong M_{pq}(\mathbb{C}), \quad (8)$$

$$\dim_{\mathbb{C}}(\mathcal{A} \vee \mathcal{B}) = (pq)^2. \quad (9)$$

The formal declarations establish (8) and (9).¹

This specialization does not formalize the direct sums in (1) and (2), the possibility of zero block-pair representations, or the multiplicity identity (3). Even in the single-block case, it does not prove the corresponding ambient representation multiplicity r or the equation

$$rpq = |I|. \quad (10)$$

Thus the matrix size pq describes the product algebra, not necessarily the whole ambient matrix algebra.

3 Why the specialization is useful

In the adjacent-support-algebra argument, the immediate use of Proposition `Cscom` is that two commuting full matrix factors span a copy of $M_{pq}(\mathbb{C})$; see lines 1270–1308 of Ref. [GNVW12]. Showing that this copy fills the adjacent physical algebra is a separate surjectivity argument. The specialization (8) supplies the first statement without claiming the second.

4 Elimination plan

Removing the restriction requires a finite direct-sum form of the multiplication representation. For every pair (μ, ν) , restrict multiplication to $M_{n(\mu)}(\mathbb{C}) \otimes M_{m(\nu)}(\mathbb{C})$, distinguish the zero and nonzero representations, and identify each nonzero image with $M_{n(\mu)m(\nu)}(\mathbb{C})$. Decompose $\mathcal{H}$ into the corresponding isotypic representation spaces. Their multiplicities then give $r_{\mu\nu}$, and summing their dimensions proves (3).

Until that construction is formalized, source-labelled statements must mark the full-matrix, single-block restriction and must not claim the multiplicity equation.

References

- [GNVW12] David Gross, Vincent Nesme, Héctor Vogts, and Reinhard F. Werner. Index theory of one dimensional quantum walks and

¹Formal declarations: `StarSubalgebra.commutingMulMap_injective_of_equiv_matrix`, `StarSubalgebra.commutingSupEquivMatrix`, and `StarSubalgebra.finrank_sup_of_equiv_matrix` in [TNLean/Algebra/CommutingStarSubalgebraProduct](#).

cellular automata. *Communications in Mathematical Physics*, 310(2):419–454, 2012.

- [SW04] Benjamin Schumacher and Reinhard F. Werner. Reversible quantum cellular automata. Preprint, 2004.