

The Anomaly Class of the Two-Qubit $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tensors: From Detector Values to the Class

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Verdict: scope-restriction (wip).

1 The target statement

Ref. [GRLM23], subsection “MPO algebras representing $\mathbb{Z}_2 \times \mathbb{Z}_2$ ”, states that $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2^3$ and lists representatives of the classes. Its mixed type-II representative takes the value -1 on the triples $(g_1 g_2^i, g_1^j g_2, g_1^k g_2)$ with $i, j, k \in \{0, 1\}$ and 1 elsewhere, where g_1, g_2 are the two generators. Equivalently, in the coordinates $a = (a_1, a_2)$,

$$\omega_{\Pi}(a, b, c) = (-1)^{a_1 b_2 c_2}. \quad (1)$$

The source says that the values $\omega_g = \omega(g, g, g)$ in a normalized gauge, at the three elements of order two, distinguish the classes. Its table assigns to the type-II class the values

$$(\omega_a, \omega_b, \omega_{ab}) = (+1, +1, -1). \quad (2)$$

The periodic-boundary construction of the same reference represents a finite group exactly by matrix product operators built from a three-cocycle. The anomaly three-cocycle of an exact representation by normal tensors is defined from its fusion tensors in Ref. [FRBC25], in the display preceding the three-cocycle equation. Its cohomology class does not depend on the choice of fusion tensors. Neither source prints the two-qubit tensors used here.

2 The formalized statement

The two-qubit tensors M_e, M_x, M_y, M_{xy} of the condensation-defect example are a construction of this development. Their periodic operators satisfy $U_g U_h = \lambda(g, h)^L U_{gh}$ on a ring of L sites, with $\lambda(g, h) = (-1)^{g_1 h_2}$, so they form only a projective family. They equal the operators of the construction for the type-II cocycle followed by the on-site operator $Z_2^{g_1}$, the Pauli Z on the second qubit of each site. Multiplying the letter of M_g with input label i by $(-1)^{g_1 i_2}$ removes

this factor. The resulting tensors M'_g have the bond dimensions of M_g , and their periodic operators are exactly those of the construction, so they form an exact representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Their doubled-index tensors are normal, since the letters differ from those of M_g by signs.

For this exact representation, let ω be the anomaly three-cochain of any choice of fusion tensors. The formalization proves, for every $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$,

$$\omega(g, e, g) \omega(g, g, g) = \omega_{\text{II}}(g, e, g) \omega_{\text{II}}(g, g, g) = (-1)^{g_1 g_2}. \quad (3)$$

The right side of (3) is +1 at (1, 0) and (0, 1) and -1 at (1, 1). The left side is invariant under the gauge action of the fusion tensors because $g^2 = e$, and in a normalized gauge it is $\omega(g, g, g)$. So these are the values (2) of the table. In particular the anomaly class of the full group is nontrivial.¹ The values are computed on the three subgroups of order two, by restricting the family and its fusion tensors along the three embeddings of $\mathbb{Z}_2$. The undressed projective family is given no anomaly class, and the sources give it none.

3 Resolution of detector separation

The detector-separation step is now proved for arbitrary three-cocycles, without a normalization hypothesis. Two three-cocycles have the same class if and only if the three products $\omega(g, e, g) \omega(g, g, g)$ agree at the nonidentity elements. Every class has a unique representative ω_{pqr} , and their parameters add under multiplication. This gives $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{C}^\times) \cong \mathbb{Z}_2^3$. The proof normalizes an arbitrary cocycle by a coboundary, then uses one square root in $\mathbb{C}^\times$ to fix six entries; the cocycle equation and the three diagonal signs determine the remaining entries.²

Applied to (3), this proves $[\omega] = [\omega_{\text{II}}]$ for every choice of fusion tensors of the exact dressed representation.³

Verdict: detector separation resolved; the family comparison is still open. The conclusion concerns the exact dressed family M'_g . It does not assign an anomaly class to the original odd-ring projective family, nor establish transport to the even-ring or two-site-blocked original family. That family comparison remains outside this result, as tracked by issue #8041, so the restriction stays live until it is settled.

¹Formal declarations: `Z2Z2Condensation.kleinFamily_isNormalRepresentation` in `TNLean/MPS/Examples/AnomalousCondensation/AnomalousCondensationZ2Z2Exact`, and `Z2Z2Condensation.cyclicInvariant_omega_kleinFamily` and `Z2Z2Condensation.not_isTrivialGaugeClass_omega_kleinFamily` in `TNLean/MPS/Examples/AnomalousCondensation/Z2Z2AnomalyClass`.

²`TNLean.Algebra.ScalarThreeCochain.cohomologousTo_iff_klein_three_cyclicInvariants`, `TNLean.Algebra.ScalarThreeCochain.existsUnique_cohomologousTo_kleinCocycleFamily`, and `TNLean.Algebra.ScalarThreeCochain.kleinH3Equiv` in `TNLean/Algebra/KleinCocycleCompleteness`.

³`Z2Z2Condensation.omega_cohomologousTo_kleinCocycle` in `TNLean/MPS/Examples/AnomalousCondensation/Z2Z2AnomalyClass`.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.
- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.