

Weak-Hopf Parent Terms: Support Averaging and Ambient Completion

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Verdict: scope-restriction (open).

1 The source and the distinction

Garre-Rubio, Lootens and Molnár define the canonical local parent term as $h = I - P$, where P is the orthogonal projector onto the local arbitrary-boundary matrix product state support. They average h for a group representation, then describe a weak-Hopf extension. Appendix B explicitly specifies the weak-Hopf construction on P :

$$E(P) = \sum \rho(\Omega_{(1)}) P \rho(T(\Omega_{(2)})), \quad \rho = (\phi \otimes \phi) \circ \Delta.$$

The normalization stated there is $E(I) = Q := \rho(1)$, rather than $E(I) = I$.¹ Consequently

$$E(I - P) = Q - E(P), \quad I - E(P) = E(I - P) + (I - Q).$$

The literal average of the parent and the ambient complement of the averaged support are different constructions. If $E(P)$ is the orthogonal projector onto the original support, as Appendix B states, then $E(P) = P$ and the latter construction is the original parent term. The nonzeroness of this complement additionally needs $P \neq I$.

2 An elementary fusion-compatible test

The following example is a mathematical derivation, independently checked at the level of the displayed formulas; it is not a Lean weak-Hopf algebra instance or a formal certificate of the complete weak-Hopf axioms. Fix $r \geq 2$, put $K = \mathbb{C}^r$,

¹Exact source: `arXiv:2203.12563v3`, `Papers/2203.12563/REsubmission.tex`, lines 1276–1320 and 2463–2466. The canonical pulling-through objects are described in `arXiv:2204.05940v1`, Section 5.5, Theorem 5.4, and Section 6, equations (6.1)–(6.4).

$V = K \otimes K$ and $\mathcal{A} = \text{End}(V)$. For matrix units $e_{ij,kl} = |i, j\rangle\langle k, l|$, define

$$\Delta(e_{ij,kl}) = \frac{1}{r} \sum_{a,b} e_{ia,kb} \otimes e_{aj,bl}, \quad \epsilon(e_{ij,kl}) = r\delta_{ij}\delta_{kl}, \quad S(e_{ij,kl}) = e_{lk,ji}.$$

These formulas define a finite-dimensional C^* -weak Hopf algebra. Indeed, with

$$W_2|i, j\rangle = \frac{1}{\sqrt{r}} \sum_a |i, a\rangle|a, j\rangle, \quad |\Phi\rangle = \frac{1}{\sqrt{r}} \sum_a |a, a\rangle,$$

one has $W_2^*W_2 = I$ and $\Delta(x) = W_2xW_2^*$. Coassociativity inserts the same two Bell pairs in either order. The weak-unit identity is the equality of the product of the two disjoint Bell projections with $\Delta^2(1)$. The counit and weak-counit identities follow by contracting their Kronecker deltas. The counital maps are

$$\epsilon_t(e_{ij,kl}) = \delta_{kl}(E_{ij} \otimes I), \quad \epsilon_s(e_{ij,kl}) = \delta_{ij}(I \otimes E_{kl}),$$

and direct multiplication gives all three antipode identities.

The positive normalized integral and its pulling-through map are

$$\Omega = \frac{1}{r} \sum_{i,k} e_{ii,kk} = |\Phi\rangle\langle\Phi|, \quad T = S.$$

In particular, $\Delta(\Omega) = r^{-2} \sum_{i,k,a,b} e_{ia,kb} \otimes e_{ai,bk}$ is cocommutative and nondegenerate, and $(1 \otimes x)\Delta(\Omega) = (S(x) \otimes 1)\Delta(\Omega)$. The dual is again this weak Hopf algebra: for the dual basis, $f_{ij,kl} \mapsto r^{-1}e_{ik,jl}$ is a star-preserving weak-Hopf isomorphism. Its target counital subalgebra is the irreducible defining module, and W_2 identifies its weak tensor square with itself. Thus both representation categories are equivalent to Vec , with one simple tensor unit, rather than merely to a multifusion category. The unique dual character equals Ω , so the canonical character construction gives exactly the displayed normalization.

Consider the injective matrix product state and operator tensors

$$A^{ij} = E_{ij}/\sqrt{r}, \quad T^{ij,kl} = (E_{ij} \otimes E_{kl})/r.$$

Their matrices span their respective full bond matrix algebras. Define $W_n : V \rightarrow V^{\otimes n}$ by inserting $n-1$ normalized Bell pairs. The arbitrary-boundary state support is $\text{ran } W_n$. For an operator boundary B , set

$$x(B)_{(i,j),(k,l)} = r^{-1}B_{(j,l),(i,k)}.$$

The corresponding operator is $O_{T,B}^n = W_n x(B) W_n^*$ at every positive length. This length-independent bijection of boundaries proves arbitrary-boundary closedness. Its action on state boundary vectors proves the length-independent compatibility condition as well. For $B = I$, $x(B) = \Omega$ and the periodic operator is the rank-one projection onto the periodic state. The fusion and action reductions insert one normalized Bell pair; their associativity coefficients can be chosen $F = L = 1$.

At two sites the support is

$$P = Q = \Delta(1) = I_K \otimes |\Phi\rangle\langle\Phi| \otimes I_K.$$

Writing $Y = W_2^* X W_2$ yields

$$E(X) = W_2 \left(\frac{1}{r^2} \sum_{i,a,k,b} e_{ia,kb} Y e_{kb,ia} \right) W_2^* = \frac{\text{Tr}(QX)}{r^2} Q.$$

Thus $E(P) = P$, while $E(I - P) = 0$ and $\text{rank}(I - P) = r^4 - r^2 > 0$. This disproves unconditional nonzeroness for the *uncompleted average*. It is fully consistent with Appendix B read as choosing the parent $I - E(P)$. No counterexample to the phase-classification theorem is asserted.

3 The local theorem and the remaining formalization scope

Let $\mathcal{B}$ be a star-closed family of matrices preserving a subspace G . For $v \in G^\perp$ and $w \in G$,

$$\langle w, Bv \rangle = \langle B^* w, v \rangle = 0.$$

Hence both G and $G^\perp$ are invariant, and their orthogonal projections commute with every member of $\mathcal{B}$. Adjoining the ambient identity does not change invariance.

The local parent theorem derives invariance from the source's arbitrary-boundary compatibility, requires adjoint closure of the local boundary range explicitly, and applies this orthogonal-reduction argument to the canonical complementary projection.² It does not assume the desired commutation, nor does it identify a nonunital weak-Hopf average with an ambient complementary projection.

4 Unit support and normalized boundary averages

There is a sufficient positive-length criterion for the missing identity $Q_n P_n = P_n$. Let T be an operator tensor and A a state tensor, compatible with one output state boundary chosen for all positive lengths. Assume that A is injective after blocking $L_0 > 0$ sites and that a fixed operator boundary U satisfies $O_{T,U}^{L_0} = I$.

²`MPOTensor.IsBoundaryCompatible.groundSpaceES_mem_invSubmodule`, `MPOTensor.IsBoundaryCompatible.groundSpaceES_invariant_boundaryStarAlgebra`, `MPOTensor.IsBoundaryCompatible.parentInteractionES_commute_mpoWithBoundary`, and `MPOTensor.IsBoundaryCompatible.parentInteraction_commute_mpoWithBoundary`, in `TNLean/MPS/Symmetry/BoundaryParentCommutation`.

Write $\Gamma_k(X) = \psi_{A,X}^k$ and $Q_n = O_{T,U}^n$. Compatibility gives one state boundary Z with

$$O_{T,U}^k \Gamma_k(X) = \Gamma_k(Z)$$

for all $k > 0$. At $k = L_0$ this becomes $\Gamma_{L_0}(Z) = \Gamma_{L_0}(X)$, so injectivity forces $Z = X$. Consequently Q_n fixes every vector of $\mathcal{S}_A^n$ and $Q_n P_n = P_n$ for every $n > 0$. The target length can be smaller than L_0 , and no adjoint closure is needed for this one-sided identity. One-site injectivity and $O_{T,U}^1 = I$ are a special case.³

Now also assume adjoint closure of the entire boundary range at the chosen target length $n > 0$. Orthogonal reduction gives $P_n O_{T,Y}^n = O_{T,Y}^n P_n$ for every boundary Y . Let $(X_j, Y_j)_{j \in J}$ be finite actual boundary families with

$$\sum_{j \in J} O_{T,X_j}^n O_{T,Y_j}^n = Q_n, \quad E_n(B) = \sum_{j \in J} O_{T,X_j}^n B O_{T,Y_j}^n.$$

This is the product normalization used in Appendix B. The derived commutation and unit-support identities imply

$$E_n(P_n) = \sum_{j \in J} (O_{T,X_j}^n O_{T,Y_j}^n) P_n = Q_n P_n = P_n.$$

It follows that

$$E_n(I - P_n) = Q_n - P_n, \quad I - E_n(P_n) = I - P_n.$$

The ambient complement is nonzero whenever $d^n > D^2$, by the dimension bound $\dim \mathcal{S}_A^n \leq D^2$. The literal parent average can still vanish when $Q_n = P_n$. The finite index set may be empty and the dimensions may be zero whenever the displayed assumptions hold. No reverse product normalization, paired-adjoint identity, or assumed value of $E_n(P_n)$ is required.⁴

5 Remaining weak-Hopf construction

The source-labelled weak-Hopf assertion remains separate. Its proof requires the C^* -weak-Hopf representation, its boundary adjoint closure, a represented unit satisfying the unit-support condition, and normalized finite boundary families from the canonical integral. The positive injectivity and identity hypotheses above are not derived from general boundary compatibility.

The periodic-parent results supply tensor cuts and cyclic-window transport under explicit local adjoint closure. Wrapping windows require the global boundary to commute with each tensor letter. Summand selectors of assembled block tensors satisfy this condition and extract the labelled periodic operators. These results supply neither the missing weak-Hopf realization nor its integral normalization.

³The boundary-vector, whole-support, and canonical-projection versions, including one-site specializations, are proved in `TNLean/MPS/Symmetry/BoundaryUnitSupport`.

⁴The commutation, support-average, literal parent-average, ambient complement, and nonzero-complement results are proved in `TNLean/MPS/Symmetry/BoundaryProjectionAverage`.