

The L-Symbol Relation Imposed by Time Reversal

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Verdict: local-correction (resolved).

1 The printed statement

Ref. [GRLM23], subsection “Interplay between MPO and time reversal symmetry” (lines 865–958 of the source), considers a matrix product state with blocks x that is symmetric under a group matrix product operator symmetry $(O_g)_{g \in G}$ and under an antiunitary time reversal $\mathcal{T} = v^{\otimes n} \mathcal{K}$ commuting with every O_g . Write $V_{g,x}$ for the action tensor mapping block x to block $g \cdot x$ and $L_{g,h}^x$ for the L-symbols defined by

$$V_{g,h \cdot x} V_{h,x} = L_{g,h}^x W_{g,h} V_{gh,x}, \quad (1)$$

the defining relation of the L-symbols in Ref. [GRLM23], written schematically with the fusion tensor $W_{g,h}$.¹ Comparing the local actions of $\mathcal{T}O_g$ and $O_g\mathcal{T}$, the source finds nonzero scalars $\gamma_{g,x}$ with $S_{g \cdot x}^{-1} V_{g,x} S_x = \gamma_{g,x} V_{g,x}^*$, and from them the display at lines 935–952,

$$\gamma_{h,x} \gamma_{g,h \cdot x} (V_{g,h \cdot x} V_{h,x})^* = L_{g,h}^x \gamma_{gh,x} (W_{g,h} V_{gh,x})^*. \quad (2)$$

The next displayed equation of Ref. [GRLM23] then states²

$$|L_{g,h}^x|^2 = \frac{\gamma_{gh,x}}{\gamma_{h,x} \gamma_{g,h \cdot x}}, \quad (3)$$

and the text concludes that the L-symbols squared are trivial because they can be written as a gauge transformation.

2 The printed relation fails

Rescaling the action tensors $V_{g,x} \mapsto c_{g,x} V_{g,x}$ multiplies $L_{g,h}^x$ by $c_{g,h \cdot x} c_{h,x} / c_{gh,x}$ and $\gamma_{g,x}$ by the unimodular number $c_{g,x} / \overline{c_{g,x}}$. The left side of (3) changes by

¹Source traceability: `Papers/2203.12563/REsubmission.tex`, line 703.

²Source traceability: `Papers/2203.12563/REsubmission.tex`, line 954.

a positive factor that can take any value, while the right side changes by a unimodular factor, so (3) cannot hold in every gauge.

An explicit instance with an on-site symmetry, where $W_{g,h}$ is trivial and there is one block, is the following. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, a, b, ab\}$ and let A be the real matrix product state tensor with components $\mathbb{K}, X, iY, Z$, where X, Y, Z are the Pauli matrices. It is invariant under $\mathcal{T} = \mathcal{K}$ with $S = \mathbb{K}$, and under the real diagonal on-site symmetry u_g whose virtual action is conjugation by the action tensors

$$V_e = \mathbb{K}, \quad V_a = X, \quad V_b = Z, \quad V_{ab} = Y. \quad (4)$$

Since V_e, V_a, V_b are real and $Y^* = -Y$, the scalars γ are $\gamma_e = \gamma_a = \gamma_b = 1$ and $\gamma_{ab} = -1$. From (4) and $XZ = -iY$ one has $L_{a,b} = -i$. Then $|L_{a,b}|^2 = 1$, while $\gamma_{ab}/(\gamma_b\gamma_a) = -1$. The numerical check with $V_b = e^{i\theta}Z$ gives $|L_{a,b}|^2 = 1$ against the ratio $-e^{-2i\theta}$.

3 The corrected statement

Taking the complex conjugate of (1) gives $(V_{g,h \cdot x} V_{h,x})^* = \overline{L_{g,h}^x} (W_{g,h} V_{gh,x})^*$. Substituting into (2) and cancelling the nonzero tensor $(W_{g,h} V_{gh,x})^*$ yields

$$\frac{\overline{L_{g,h}^x}}{L_{g,h}^x} = \frac{\gamma_{gh,x}}{\gamma_{h,x} \gamma_{g,h \cdot x}}. \quad (5)$$

In the example, $\overline{L_{a,b}}/L_{a,b} = i/(-i) = -1$, in agreement with (5). Since $\overline{L}/L = (|L|/L)^2$, the relation (5) says that the square of the phase $L/|L|$ is a gauge transformation of the trivial L-symbol; this is the correct reading of the statement that the L-symbols squared are trivial. The modulus $|L|$ is not constrained. The printed (3) replaces the ratio $\overline{L}/L$ by $\overline{L}L$.

For a general matrix product operator symmetry the fusion tensors also pick up time-reversal phases $\beta_{g,h}$, as at lines 873–889 of the source, which yield $\overline{\omega} = \beta \triangleright \omega$. The source's relation for L omits β : it is the case $\overline{L} = (1, \gamma) \triangleright L$ of the joint gauge. The same- β pair, $\overline{\omega} = \beta \triangleright \omega$ together with $\overline{L} = (\beta, \gamma) \triangleright L$, is the gauge-consistent generalization, but it is not derived in the source and is not derived here from a time-reversal operator. The formalization treats it as a hypothesis: under these two relations a single joint gauge makes both ω and L real, by taking unimodular square roots of the phases of β and γ . This is a conditional statement beyond the source.

4 Formalized statement

The formalization proves, at the level of scalars, the cancellation step from (2) and the conjugate of (1) to (5); the statement that the squared phase of L is the gauge ratio; the reality of the anomaly under lines 873–891; and, as a conditional statement under the same- β hypothesis, the joint reality of ω and

L .³ The antiunitary symmetry itself, and the derivation of γ and β from $\mathcal{T}$ commuting with O_g , are not formalized.

Verdict: resolved local correction. The printed relation (3) is false; the source’s own preceding display gives (5), which is the relation used in the formalization, and the source’s conclusion that the L-symbols squared are trivial holds in that corrected form.

References

- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.

³Formal declarations: `TNLean.Algebra.LSymbol.star_apply_eq_gauge_one_apply`, `TNLean.Algebra.LSymbol.phase_sq_eq_of_gauge_eq_star`, `TNLean.Algebra.ScalarThreeCoChain.exists_cohomologousTo_star_eq_self` and `TNLean.Algebra.LSymbol.exists_gauge_star_eq_self` in `TNLean/Algebra/ScalarThreeCocycleTimeReversal`.