

The $\mathbb{Z}_3$ Phase of $\text{Rep}(S_3)$: the Printed Action of ψ

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Verdict: false-source (resolved).

1 The source claim

Garre-Rubio, Lootens and Molnár list the phases of matrix product states symmetric under matrix product operators representing $\text{Rep}(S_3)$, with labels $1, \pi, \psi$ and fusion rules $\pi \times \pi = 1 + \pi + \psi$, $\psi \times \psi = 1$, $\psi \times \pi = \pi \times \psi = \pi$, and print the action of the labels on the blocks of each phase [GRLM23]. With N_{ab}^c the multiplicity of c in $a \times b$, the multiplicities $M_{a,x}^y$ of the action of a on a block x satisfy

$$\sum_c N_{ab}^c M_{c,x}^y = \sum_z M_{b,x}^z M_{a,z}^y, \quad (1)$$

a consequence of the associativity of the action.¹

2 The printed action

The printed $\mathbb{Z}_3$ action is $\pi \cdot x = y + z$, $\pi \cdot y = x + z$, $\pi \cdot z = x + y$ and $\psi \cdot x = x$, $\psi \cdot y = z$, $\psi \cdot z = y$.² It does not satisfy (1) for $\pi \times \pi = 1 + \pi + \psi$: on the block y , $\pi \cdot (\pi \cdot y) = x + 2y + z$, while $(1 + \pi + \psi) \cdot y = x + y + 2z$. With ψ fixing every block, (1) holds. This corrected action is the one the source itself describes when it identifies the phase with the blocks x, y, z of $\mathcal{M}_{TY}$ of $su(2)_4$, along the embedding $1, \pi, \psi \mapsto 0, 1, 2$, and the label 2 of $su(2)_4$ fixes every block of $\mathcal{M}_{TY}$.³ It is also the action that the representation theory predicts: the blocks are the three characters of $\mathbb{Z}_3$, and ψ acts by tensoring with the restriction of the sign

¹Source traceability: `Papers/2203.12563/REsubmission.tex`, lines 564–565 for (1), line 1934 for the fusion rules.

²Source traceability: lines 1959–1971.

³Source traceability: lines 1911–1925 for the action on $\mathcal{M}_{TY}$, line 1934 for the embedding, line 1972 for the identification.

representation to $\mathbb{Z}_3$, which is trivial. The formal $\mathbb{Z}_3$ action is the corrected one. That the printed one fails (1) is proved separately.⁴

3 Verdict

The printed $\mathbb{Z}_3$ action is false as printed: it does not satisfy the relation (1) that every action on blocks satisfies. The correction changes only the ψ row and is forced by the source's own identification of the phase. The remaining claims about this phase are unaffected.

References

- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.

⁴Results: `MP0Tensor.isNIMRep_repS3Z3`, `MP0Tensor.not_isNIMRep_repS3Z3Printed` and `MP0Tensor.su24TY_repS3ToSU24_castSucc`.