

The $\text{Rep}(S_3)$ and $su(2)_4$ Phases: Completeness of the Lists

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Verdict: scope-restriction (open).

1 The source claims

Garre-Rubio, Lootens and Molnár list the phases of matrix product states symmetric under matrix product operators representing $su(2)_4$ and $\text{Rep}(S_3)$ [GRLM23]. For $su(2)_4$, with labels $0, \frac{1}{2}, 1, \frac{3}{2}, 2$, they state that there are only two phases, the one with five blocks acted on by the fusion rules and the module $\mathcal{M}_{TY}$ with four blocks x, y, z, s , and they print the action of every label on the blocks of $\mathcal{M}_{TY}$.¹ For $\text{Rep}(S_3)$, with labels $1, \pi, \psi$, they state that every phase is characterized by a subgroup of S_3 , and print the actions on the blocks for $H = \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, S_3$. They identify the $\mathbb{Z}_1, \mathbb{Z}_2$ and $\mathbb{Z}_3$ phases with restrictions of the $su(2)_4$ phases along the embedding $1, \pi, \psi \mapsto 0, 1, 2$: the block s of $\mathcal{M}_{TY}$, the blocks $\frac{1}{2}, \frac{3}{2}$ of the first phase, and the blocks x, y, z of $\mathcal{M}_{TY}$.² Write N_{ab}^c for the multiplicity of c in $a \times b$. The multiplicities $M_{a,x}^y$ of each phase form a nonnegative integer representation of the fusion ring,

$$\sum_c N_{ab}^c M_{c,x}^y = \sum_z M_{b,x}^z M_{a,z}^y, \quad (1)$$

with the unit acting as the identity.

2 Completeness of the lists

Formalized: every listed action satisfies (1), with the $\mathbb{Z}_3$ action corrected as recorded in the note on the $\mathbb{Z}_3$ table, the restrictions stated by the source hold, and on a single block the list is complete. A representation on one block with the unit acting trivially is a ring homomorphism to $\mathbb{Z}$ that is nonnegative on the labels; for $\text{Rep}(S_3)$ the only one is $(1, 2, 1)$, the $\mathbb{Z}_1$ phase, and $su(2)_4$ has none, since it would force $m_2 = 1$, $m_1 = 2$ and $m_{1/2}^2 = 3$.³

¹Source traceability: `Papers/2203.12563/RSubmission.tex`, lines 1890–1931.

²Source traceability: lines 1933–1989; the restrictions at lines 1927, 1940, 1957 and 1972.

³Results: `MP0Tensor.isFusionCharacter_repS3Fusion_iff` and `MP0Tensor.not_isFusionCharacter_su24Fusion`.

Not formalized: completeness of the lists beyond one block. The source takes it from the classification of module categories, which it cites. It is not a consequence of (1) with trivial unit action alone: for $\text{Rep}(S_3)$ on two blocks,

$$M_\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_\pi = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \quad (2)$$

satisfies (1), but the representation (2) is neither the $\mathbb{Z}_2$ phase nor a sum of two copies of the $\mathbb{Z}_1$ phase. It violates the duality $M_{a^*} = M_a^\top$ of a module category, as π is self-dual.

Elimination plan: formalize the duality condition on representations; bound the number of blocks of an indecomposable representation of each ring satisfying it; and enumerate the finitely many candidates, as is done for the Fibonacci ring on two blocks.

References

- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.