

Typed Multiplicity Indices for the Coupled Pentagon

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Verdict: local-correction (wip).

The action map of Ref. [GRLM23] has type $V_{ax}^{y,i} : \mathbb{C}^{\chi_a} \otimes \mathbb{C}^{D_x} \rightarrow \mathbb{C}^{D_y}$, while $\widehat{V}$ is its synthesis partner. Consequently the displayed change of action trees is $H_{\text{seq}} = LH_{\text{fus}}$ and its coefficient contraction is $L \otimes I = H_{\text{seq}} S_{\text{fus}}$. The fusion change uses the opposite tree direction, $H_{\text{left}} = FH_{\text{right}}$. These orientations follow from the typed maps and the diagram legs, without adjoints or transposes.¹

In the coupled pentagon, the surrounding action factors fix the ranges

$$\eta \in \{1, \dots, N_{ab}^d\}, \quad \chi \in \{1, \dots, N_{dc}^e\}, \quad (1)$$

$$\mu \in \{1, \dots, N_{bc}^f\}, \quad \nu \in \{1, \dots, N_{af}^e\}. \quad (2)$$

The printed factor $(F_{abc}^e)^{d,\chi\eta}_{f,\nu\mu}$ reverses both pairs. It need not be well typed when these multiplicities differ. The typed coupled pentagon is

$$\sum_{d,n,\eta,\chi} (L_{abt}^y)^{z,ij}_{d,n\eta} (L_{dcx}^y)^{t,nk}_{e,m\chi} (F_{abc}^e)^{d,\eta\chi}_{f,\mu\nu} = \sum_l (L_{bcx}^z)^{t,jk}_{f,l\mu} (L_{afx}^y)^{z,il}_{e,m\nu}. \quad (3)$$

Only the two index orders in the F factor differ from the source's `coupledpent`, at lines 554–562.

In the following fusion gauge formula, use fresh labels: a, b, c for the three input blocks, d for the final block, and e, f for the intermediate blocks. For analysis-basis gauges $H'_{\text{left}} = Y_{\text{left}} H_{\text{left}}$ and $H'_{\text{right}} = Y_{\text{right}} H_{\text{right}}$, the fusion law is

$$F' = Y_{\text{left}} F Y_{\text{right}}^{-1}, \quad L' = X_{\text{seq}} L X_{\text{fus}}^{-1}. \quad (4)$$

Thus an inverse gauge on the right has its old coordinate as row and its new coordinate as column. The component display at lines 417–422 also needs primed indices on the F entry being summed. Its well-typed entry is

$$(F')_{f,\chi\eta}^{e,\mu\nu} = \sum_{\mu',\nu',\chi',\eta'} (Y_{ab}^e)_{\mu\mu'} (Y_{ec}^d)_{\nu\nu'} F_{f,\chi'\eta'}^{e,\mu'\nu'} (Y_{bc}^f)_{\chi'\chi}^{-1} (Y_{af}^d)_{\eta'\eta}^{-1}. \quad (5)$$

¹Source traceability: `Papers/2203.12563/REsubmission.tex`, action-map type at line 459, diagram definitions at lines 97–159, and labels `Fsymbolsdef`, `eq:F.symbol12`, and `1Fsymbol`.

These are ordinary inverse matrices acting on dual coordinate positions, not adjoints. The source’s phrase “non-zero constants” means an invertible change of basis; individual matrix entries may be zero.

The index correction follows from the displayed domains. The multiplicity L matrices, their inverse identities, and their defining analysis-tree equation are formalized.² A proof implementation of the corrected coupled pentagon (3) now derives it from the exact fusion and action maps by triangular physical padding and a common positive blocking. Its coefficients are those of the original unblocked maps, and no coherence equation is a premise.³ The general multiplicity-gauge law remains a separate proof obligation; it is not assumed by either matrix construction.

References

- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.

²Formal declarations: `MPOTensor.actionLMatrix_cross`, `MPOTensor.actionLMatrix_inverse_of_isInjective`, and `MPOTensor.actionLMatrix_analysis`, in `TNLean/MPS/MPDO/BoundaryActionLMatrix`.

³New proof implementations: `MPOTensor.actionLMatrix_mixed_pentagon` and `MPOTensor.IsBoundaryCompatible.exists_exact_mixed_pentagon`, in `TNLean/MPS/MPDO/BoundarySourceMixedPentagon`.