

Matrix Product Operator Symmetric States: Formalized Layers and Remaining Scope

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Verdict: scope-restriction (open).

1 The source definitions

Garre-Rubio, Lootens and Molnár define a matrix product operator algebra through the closedness of the arbitrary-boundary operators $\mathcal{A}_T^n = \{O_{T,B}^n : B \in \mathcal{M}_\chi\}$ under multiplication, and a symmetric matrix product state through the invariance $\mathcal{A}_T \cdot \mathcal{S}_A \subset \mathcal{S}_A$ of the arbitrary-boundary subspace $\mathcal{S}_A^n = \{|\psi_{A,X}^n\rangle : X \in \mathcal{M}_D\}$ for every system size n [GRLM23].¹ The periodic operators $O_a = O_{T_a, \mathbb{K}}$ of the blocks a then satisfy

$$O_a O_b = \sum_c N_{ab}^c O_c \quad (1)$$

for every system size, with $N_{ab}^c \in \mathbb{N}$, and the action of a block a on a block x of the state decomposes into blocks y with multiplicities $M_{a,x}^y \in \mathbb{N}$ through the action tensors. Associativity of the action gives

$$\sum_c N_{ab}^c M_{c,x}^y = \sum_z M_{b,x}^z M_{a,z}^y. \quad (2)$$

² Evaluating the invariance with periodic boundaries, the source obtains

$$O_a |\psi_{A_x}\rangle = \sum_y M_{a,x}^y |\psi_{A_y}\rangle \quad (3)$$

for general fusion rules, and recalls this form when it constructs boundary states.³ For groups the source also formulates a weaker, periodic-boundary version of both conditions, in which the boundaries are proportional to the identity in each block.⁴

¹Source traceability: `Papers/2203.12563/REsubmission.tex`, lines 311–345 (the subspace $\mathcal{S}_A^n$ and the closedness condition) and lines 431–434 (the definition of a symmetric state).

²Source traceability: line 361 for (1); lines 458–460 for the multiplicities; lines 491–492 for associativity; lines 564–565 for (2); line 683 for the group case $M_{gh,x}^y = \sum_z M_{g,z}^y M_{h,x}^z$.

³Source traceability: lines 567–568 and line 1801.

⁴Source traceability: lines 994–1000 and 1062–1064.

2 What is formalized

The formalization has several layers. They have different hypotheses and must not be conflated with the arbitrary-boundary existence theorem.

Uniform arbitrary-boundary transport. The closedness and compatibility predicates have the order $\forall X, Y \exists Z \forall N \geq 1$; one output boundary works for all lengths. Stacking sends (X, Y) to $X \otimes Y$. Choosing output boundaries on the matrix-unit basis and extending linearly proves that each predicate is equivalent to one linear boundary map on the entire stacked matrix space, independent of the positive length. This proves the first assertion of Appendix A without assuming linearity of an arbitrary choice of output boundary.⁵ For supplied exact decompositions $B^i = \sum_c W_c A_c^i V_c$ with $V_c W_b = \delta_{cb} 1$, the transported boundaries are $V_c X W_c$ at every positive length. The empty-word identity additionally requires $\sum_c W_c V_c = 1$ on the ambient bond space. These forward formulas do not construct the decomposition from closedness.⁶

Exact arbitrary-boundary reconstruction. Merged PR #8689 supplies the source-exact reconstruction. The assembled tensors are literal unweighted block sums. For fusion, the operator blocks are individually injective on the doubled physical alphabet, have positive virtual dimensions, and have no scalar-gauge duplicates. For the action construction, these target-block requirements concern the state blocks; no injectivity or separation of the incoming operator blocks is needed for that existence theorem. A scalar gauge here is an invertible similarity together with any nonzero complex scalar.⁷

The common positive word-spanning length is derived from these individual block hypotheses. The trace-dual boundary map becomes a non-unital algebra representation by comparing that length with twice that length. Matrix-unit splitting then gives biorthogonal analysis and synthesis matrices and reconstructs the original unblocked letters exactly, with no nilpotent remainder. Their synthesis-analysis sum is the represented identity, which may be a proper support idempotent. No adjoint preservation, star closure, or ambient identity resolution is assumed.⁸ This exact representation splitting discharges the reconstruction step of Appendix A; it is not a separate assertion that every arbitrary two-sided linear-map expansion has a globally minimal number of summands.

⁵`MPOTensor.IsBoundaryClosed`, `MPOTensor.IsBoundaryCompatible`, `MPOTensor.isBoundaryClosed_iff_exists_linearMap`, and `MPOTensor.isBoundaryCompatible_iff_exists_linearMap`, in `TNLean/MPS/MPDO/BoundaryClosedness`.

⁶`MPSTensor.IsBiorthogonalDecomposition.mpvWithBoundary` and `MPSTensor.IsBiorthogonalDecomposition.evalWord_of_complete`, in `TNLean/MPS/MPDO/BoundaryTransport`.

⁷`MPOTensor.IsBoundaryClosed.exists_blockFusionDecomposition_of_isInjective` and `MPOTensor.IsBoundaryCompatible.exists_blockActionDecomposition_of_isInjective`, in `TNLean/MPS/MPDO/BoundarySourceDecomposition`.

⁸`MPSTensor.exists_positive_wordTupleSpanTop_of_isInjective`, in `TNLean/MPS/MPDO/InjectiveBlockWordSpan`; `Matrix.exists_piMatrix_blocks`, in `TNLean/Algebra/PiMatrixRepresentation`; and `MPSTensor.exists_biorthogonalDecomposition_of_boundaryTransport`, in `TNLean/MPS/MPDO/BoundaryBiorthogonal`.

Taking identity boundaries in the constructed exact maps gives the periodic fusion and action equations, integer multiplicities, their NIM representation law, and fusion associativity. The simultaneous state independence used in the proof is derived from the same individual block hypotheses, rather than added to the source theorem.⁹

Periodic fusion and action. The periodic fusion hypothesis is (1) at every positive length. The periodic symmetry hypothesis is (3), initially with complex coefficients independent of length. For normal state blocks whose periodic vectors are linearly independent at one positive length, the formalization proves that these coefficients are nonnegative integers and satisfy (2). The source obtains integrality instead from its exact action tensors. Its standing assumption that distinct blocks become orthogonal in the thermodynamic limit supplies eventual linear independence; it is not to be replaced by a new unrelated independence assumption in the source theorem.¹⁰ The predicate for a single symmetric block does not itself require that all its action coefficients are one; that extra clause is essential to the source’s single-block obstruction at line 610.

Fusion and action reductions with multiplicity. The periodic identities produce mutually biorthogonal fusion and action reductions, one pair for every copy of each output channel. They reconstruct the stacked tensor only up to a nilpotent remainder.¹¹ This is not the source’s exact letterwise decomposition from arbitrary-boundary closedness and invariance. For a single normal target, two reductions are proportional after multiplication by sufficiently long words. For several targets, periodic data alone does not force a pure change of basis in the multiplicity spaces: upper-triangular changes can mix the chosen maps with residual directions.

When an exact biorthogonal zipper decomposition is supplied, uniqueness up to an invertible matrix on each multiplicity space is proved.¹² The exact source reconstructions also admit unique invertible multiplicity gauges on their letter support under the stated target-block hypotheses.¹³ These comparison maps do not by themselves prove the joint L-symbol gauge law, and they are not an assertion about unrestricted periodic reductions.

⁹`MPOTensor.IsBoundaryClosed.exists_isNIMRep_of_isBoundaryCompatible`, in `TNLean/MPS/MPDO/BoundaryMultiplicity`.

¹⁰Formal declarations: `MPOTensor.IsMPOFusionAlgebra`, `MPOTensor.IsMPOSymmetricFamily`, `MPOTensor.IsMPOSymmetric`, and `MPOTensor.exists_isNIMRep_of_isMPOSymmetricFamily`. The integer-coefficient action of an invertible label permutes the blocks, as expressed by `MPOTensor.exists_equiv_mpo_mulVec_eq_of_isInvertibleLabel`.

¹¹`MPOTensor.exists_fusionTensors_of_mpo_mul_eq_sum` and `MPOTensor.exists_actionTensors_of_mpo_mulVec_eq_sum`, in `TNLean/MPS/Symmetry/MPOSymmetry/FusionTensors`.

¹²`MPOTensor.exists_unique_zipperFusionGauge` and `MPOTensor.exists_unique_zipperActionGauge`, in `TNLean/MPS/Symmetry/MPOSymmetry/ZipperUniqueness`.

¹³`MPSTensor.IsBiorthogonalDecomposition.exists_unique_multiplicityGauge_of_isInjective`, in `TNLean/MPS/MPDO/BoundaryZipperUniqueness`.

Multiplicity F-symbols for complete zipper families. Complete zipper fusion families carry independently specified synthesis and analysis maps, their biorthogonality and exact reconstruction identities, and a simultaneous inverse separating the labelled block letters. Their support projector need not be the identity on the full product bond space. For such a family the formalization constructs the invertible F-matrix, proves the F-move and pentagon, and proves covariance under the full multiplicity gauge.¹⁴ The printed F-matrix and its inverse have opposite orientations; the pentagon must use the stated orientation. Constructions from a compression with zero remainder and from a star-closed word algebra are also available. Those compression and star-closed-algebra routes are distinct from the source-exact reconstruction above. The latter also supplies a complete zipper family after a derived common positive physical blocking, with the original exact virtual maps and multiplicities unchanged. A simultaneous one-letter inverse at the unblocked alphabet is not imposed.¹⁵ Star closure is not a standing hypothesis of the source’s Sections 2–3.

Multiplicity L matrices and the coupled pentagon. For the exact fusion and action maps, sequential analysis followed by fusion-then-action synthesis determines the multiplicity L matrix. For a fixed final block y , each coefficient satisfies

$$(H_{\text{seq}})_p(S_{\text{fus}})_q = L_{pq}I_{D_y}. \quad (4)$$

Normalized trace extracts L_{pq} . Positive-dimensional, individually injective, scalar-gauge-separated state blocks give its inverse identities without a supplied simultaneous-span premise. The collected analysis-tree identity is also proved; its intermediate common-span hypothesis is supplied by those same block assumptions.¹⁶

The current draft PR #8695 derives the multiplicity coupled pentagon for these actual coefficients from exact tree comparisons. Its source existence theorem uses the uniform arbitrary-boundary closedness and compatibility conditions above, unweighted block assemblies, and the individual injectivity, positive dimensions, and scalar-gauge separation of both the operator and state families. It assumes no coherence equation. Triangular physical padding and a common positive blocking leave the original virtual maps, and hence their coefficients,

¹⁴`MPOTensor.CompleteZipperFusionFamily`, `MPOTensor.CompleteZipperFusionFamily.printedFMatrix`, `MPOTensor.CompleteZipperFusionFamily.rightTripleSynthesis_mul_printedFMatrix`, `MPOTensor.CompleteZipperFusionFamily.inversePrintedFMatrix_pentagon`, and `MPOTensor.CompleteZipperFusionFamily.printedFMatrix_regauge`, in `TNLean/MPS/MPDO/CompleteZipperFusionDefs`, `TNLean/MPS/MPDO/CompleteZipperFusion`, `TNLean/MPS/MPDO/CompleteZipperFusionPentagon`, and `TNLean/MPS/MPDO/CompleteZipperFusionGauge`.

¹⁵`MPOTensor.IsBoundaryClosed.exists_completeZipperFusionFamily`, in `TNLean/MPS/MPDO/BoundaryZipperBlocked`.

¹⁶`MPOTensor.actionLMatrix_cross`, `MPOTensor.actionLMatrix_inverse_of_isInjective`, and `MPOTensor.actionLMatrix_analysis`, in `TNLean/MPS/MPDO/BoundaryActionLMatrix`.

unchanged.¹⁷ The fusion coefficient uses left analysis followed by right synthesis, so it is the inverse printed F-matrix of the complete-zipper convention. The coupled F factor has row (d, η, χ) and column (f, μ, ν) , correcting the source's reversed multiplicity pairs as documented in `docs/paper-gaps/glm23_multiplicity_1_indices.tex`. Individual L entries may vanish. These results do not establish the general joint multiplicity gauge law.

Group cocycles, L-symbols, and the group obstruction. For a normal periodic group representation, the two fusion trees define a nonzero boundary-dressed associator scalar. It is a three-cocycle; changing the fusion tensors changes it by a coboundary. Its class is expressed in Mathlib's third group cohomology.¹⁸ For an action permuting normal state blocks, the fusion and action reductions similarly construct scalar L-symbols and prove their coupled pentagon with that associator. Restriction to a fixed block trivializes the anomaly on its stabilizer. In particular, a nontrivial group anomaly excludes an invariant normal single-block state with action multiplicity one. This group cohomological obstruction is already formalized; it must not be described as absent.¹⁹ The scalar gauge operation and its preservation of compatibility are proved. Deriving the full L-symbol gauge law from every change of fusion and action tensors remains distinct from these scalar identities.

For a fixed transitive group action, the scalar reconstruction formula and the torsor of action-gauge solution classes under $H^2(H, \mathbb{C}^\times)$ are proved. The formula is first used in a fusion gauge in which the anomaly is identically one on the stabilizer and then transported back; the required gauge exists whenever the restricted class is trivial. The correction is recorded in [con26]. This is an algebraic classification of scalar solutions with the fusion gauge held fixed, not the physical Hamiltonian-path classification.

The source's lemma that exact closed fusion and $U_e = \mathbb{K}$ force an injective group representation to be on-site, with trivial anomaly, is also proved from the exact fusion identities. The general exact reconstruction above does not yet specialize its maps to the one-channel closed-group data required by these declarations; that adapter is distinct from both the completed reconstruction and the periodic, boundary-dressed group reductions.²⁰

¹⁷`MPOTensor.actionLMatrix_mixed_pentagon` and `MPOTensor.IsBoundaryCompatible.exists_exact_mixed_pentagon`, in `TNLean/MPS/MPDO/BoundarySourceMixedPentagon`.

¹⁸`MPOTensor.GroupFamily.FusionData.omega`, `MPOTensor.GroupFamily.FusionData.isCocycle_omega`, `MPOTensor.GroupFamily.FusionData.omega_eq_fusionGauge`, and `MPOTensor.GroupFamily.IsNormalRepresentation.anomalyClass`, in `TNLean/MPS/Symmetry/MPOSymmetry/Associator` and `TNLean/MPS/Symmetry/MPOSymmetry/AssociatorCohomology`.

¹⁹`MPOTensor.GroupFamily.BlockActionData.lSymbol`, `MPOTensor.GroupFamily.BlockActionData.isCompatible_lSymbol`, `MPOTensor.GroupFamily.IsNormalRepresentation.isTrivialGaugeClass_comap_omega_of_fixed`, and `MPOTensor.GroupFamily.IsNormalRepresentation.not_exists_invariant_of_not_isTrivialGaugeClass`, in `TNLean/MPS/Symmetry/MPOSymmetry/PermutedBlocks` and `TNLean/MPS/Symmetry/MPOSymmetry/AnomalyObstruction`.

²⁰`MPOTensor.GroupFamily.isOnSite_of_closed_fusion` and `MPOTensor.GroupFamily.anomalyClass_eq_zero_of_closed_fusion`, in `TNLean/MPS/Symmetry/MPOSymmetry/ClosedFusion`.

Other periodic consequences. The fusion-ring obstruction to a single normal symmetric block is also proved: its eigenvalues define a nonnegative integer fusion character. This excludes Fibonacci, but it is not the general F-symbol obstruction of source lines 610–636. Separately, for a fusion ring and a unital nonnegative integer representation, a strictly positive common eigenvector produces the symmetric boundary state of source lines 1795–1805. That construction requires neither commutativity of the fusion ring nor indecomposability of the action.²¹

3 Remaining boundary

The arbitrary-boundary contractions, the length-independent boundary transport, and the exact fusion/action reconstruction are no longer missing. In particular, the merged result chooses one output boundary before all positive lengths; merely choosing a new boundary separately at each length is not substituted for the source’s condition. The matrix-unit splitting retains the possibly nonunital support instead of adding a star-closure, vanishing-remainder, or ambient-completeness premise. The multiplicity L construction and the corrected coupled pentagon are also formalized.

The following algebraic obligations remain distinct:

1. Derive the full joint multiplicity gauge law for L from arbitrary compatible changes of the original fusion and action maps, beyond the proved complete-fusion gauge covariance and scalar group laws.
2. Prove the source’s general single-block, multiplicity-one obstruction that trivializes the F-symbols, beyond the scalar group case and the different fusion-ring obstruction.
3. Connect the exact source maps to the one-channel closed-group and coset constructions where an exact adapter is still required; periodic or boundary-dressed reductions alone are not that adapter.

The original tracking issues are GitHub issue #7968 and GitHub issue #8045; the completed reconstruction and coherence components above must be distinguished from these remaining questions. The periodic predicates remain useful as explicitly weaker hypotheses and need not be renamed as the arbitrary-boundary ones.

Further independent work concerns tensor-derived time-reversal constraints; the weak-Hopf realization, represented-unit support, and canonical-integral normalization used by the source’s parent construction; the general physical phase classification of Section 5, including non-fixed-point and varying-bond paths; and remaining exact examples and completeness claims in Section 6. The local parent and finite-boundary averaging results do not themselves supply the

²¹`MPOTensor.not_isMPOSymmetric_of_forall_not_isFusionCharacter` and `MPOTensor.exists_pos_symmetric_boundary_state`, in `TNLean/MPS/Symmetry/MPOSymmetry/Character` and `TNLean/MPS/Symmetry/MPOSymmetry/SymmetricBoundary`.

missing weak-Hopf realization and integral normalization; see `docs/paper-gaps/glm23_who_parent_completion.tex`. None of these physical conclusions follows just from the algebraic scalar classification or the multiplicity pentagon. The printed time-reversal L relation is subject to the resolved correction in `docs/paper-gaps/glm23_time_reversal_l_symbol.tex`.

4 Normality of the mixed operator

The mixed-operator discussion in `REsubmission.tex`, lines 1687–1689, asserts injectivity in the setting of aligned fusion and action data. A separate result proves injectivity after blocking three sites from fewer assumptions: positive endpoint state dimensions, injective endpoint operator tensors, and biorthogonal action maps with positive common multiplicity. It uses neither common F/L-symbols nor ambient completeness. This establishes normality of the actual mixed operator, but does not prove the source’s one-site assertion in its full categorical setting.

Indeed, a nonzero coefficient in each crossed corner follows by sandwiching the crossed action contraction between one fixed analysis and synthesis pair. The endpoint one-letter spans supply diagonal matrix units. Sandwiching a crossed coefficient between two such units then supplies every crossed matrix unit in the three-letter span; diagonal units can be padded to the same length. This argument has no one-letter crossed-span conclusion.

The one-site assertion remains a separate obligation: determine what the full aligned fusion/action hypotheses imply about the crossed one-letter spans. An example satisfying only the isolated biorthogonal assumptions cannot refute that stronger source setting. The three-site normality result is an independent sufficient input for later normal operator-family arguments, which must still verify all their other hypotheses, including physical adjoint compatibility where required.

5 Elimination plan

Use the completed exact arbitrary-boundary reconstruction and the actual multiplicity F/L comparisons as the starting point. Derive the remaining joint multiplicity gauge law with the source’s corrected dual index positions, then address the general multiplicity-one F-symbol obstruction. Complete the exact one-channel group and coset adapters without replacing exact identities by long-word, boundary-dressed ones. Keep the already proved scalar group and fusion-ring results as separate reusable inputs.

The remaining tasks concerning tensor-derived time reversal, weak-Hopf realization, physical phase classification, and example completeness retain their own source hypotheses and correction notes. Completing the algebraic reconstruction and coupled pentagon does not remove those independent proof obligations.

References

- [con26] The TNLean contributors. The fusion gauge of the stabilizer reconstruction formula. Paper-gap note glm23_eq20_fusion_gauge, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/glm23_eq20_fusion_gauge.pdf.
- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.