

The Printed One-Qubit $\mathbb{Z}_2 \times \mathbb{Z}_2$ Symmetry: Diagonal Values of the Anomaly

2026-10-10

Verdict: local-correction (resolved).

1 The source statement

Ref. [GRLM23], subsection “MPO algebras representing $\mathbb{Z}_2 \times \mathbb{Z}_2$ ”, realizes one of the classes by the operators $U_a = \prod_i CZ_{i,i+1}Z_i$, $U_b = \prod_i X_i$, and $U_{ab} = U_a U_b$ on a periodic chain, and states: “In this realization the only non-trivial 3-cocycle element is $\omega(ab, ab, ab) = -1$.” Read entrywise, this is impossible: a normalized cochain whose only value different from one is $\omega(ab, ab, ab) = -1$ violates the cocycle equation

$$\omega(h, k, l) \omega(g, hk, l) \omega(g, h, k) = \omega(gh, k, l) \omega(g, h, kl) \quad (1)$$

at $(g, h, k, l) = (a, b, ab, ab)$, where the left side is 1 and the right side is -1 . The source defines $\omega_g = \omega(g, g, g)$ (`Papers/2203.12563/REsubmission.tex`, line 1852), and its table row for the class $(0, 0, 1)$ (line 1865) lists $(\omega_a, \omega_b, \omega_{ab}) = (+1, +1, -1)$ with trivializing subgroups $\{H_e, H_a, H_b\}$; line 1886 places the two phases of the printed realization at $H = H_a$ and $H = H_b$. The statement is therefore read as a statement about the diagonal values. In a normalized gauge the three diagonal values determine the class, and the statement asserts

$$(\omega(a, a, a), \omega(b, b, b), \omega(ab, ab, ab)) = (+1, +1, -1). \quad (2)$$

2 What is formalized

For every choice of fusion tensors of the four printed tensors, the cyclic products $\omega(g, e, g) \omega(g, g, g)$ at $g = a, b, ab$ are $+1$, $+1$ and -1 . The value at g is computed on the restriction of the family to the subgroup $\{e, g\}$. On $\{e, ab\}$ the family is the decorated CZX family. On $\{e, b\}$ all tensors have bond dimension one and the one-by-one identity fusion tensors give the trivial cochain. On $\{e, a\}$ the stacked square of the bond-two diagonal tensor A of U_a reduces to the identity tensor through the row $\langle 00|$ and the column $|00\rangle + |11\rangle$, and the two fusion trees of three copies of A agree against every nonempty word.

The classification of Klein three-cocycles by these three cyclic products then identifies the class of ω with that of the representative of the row $(0, 0, 1)$, the type-II cocycle

$$\omega_{\text{II}}(x, y, z) = (-1)^{x_1 y_2 z_2}, \quad (3)$$

where $x = (x_1, x_2)$ with $a = (1, 0)$ and $b = (0, 1)$. Equivalently, the class corresponds to $(0, 0, 1)$ under $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{C}^\times) \cong \mathbb{Z}_2^3$, and every normalized representative has the diagonal values (2).¹

3 The L-symbol relations

The source continues (line 1886): for normalized L -symbols, in both phases $L_{ab,ab}^x = -L_{ab,ab}^y$, and $L_{a,a}^x = L_{a,a}^y$ for $H = H_a$ and $L_{b,b}^x = L_{b,b}^y$ for $H = H_b$, where x, y are the two blocks. For an involution g , the compatibility of normalized L -symbols with a normalized cocycle ν at (g, g, g) gives

$$L_{g,g}^x = \nu(g, g, g) L_{g,g}^{g \cdot x}, \quad (4)$$

so the diagonal values give $L_{ab,ab}^x = -L_{ab,ab}^{ab \cdot x}$, $L_{a,a}^x = L_{a,a}^{a \cdot x}$ and $L_{b,b}^x = L_{b,b}^{b \cdot x}$. This is what is formalized.² Since ab lies in neither H_a nor H_b , it exchanges the two blocks in both phases, which is the first printed relation. The relation for a relates two different blocks exactly when a moves x , that is, in the phase $H = H_b$; likewise the relation for b is a relation between the two blocks in the phase $H = H_a$. In the phase $H = H_a$ the element a fixes both blocks, and the action-tensor gauge with $\gamma_a(x) = i$ and $\gamma_g(z) = 1$ otherwise keeps the L -symbols normalized and compatible while changing the sign of $L_{a,a}^x$ but not of $L_{a,a}^y$. The printed relation for a in the phase $H = H_a$ therefore holds for a particular solution, not for every normalized one; the gauge-invariant relation is (4).

Verdict: local correction, resolved. The printed sentence on $\omega(ab, ab, ab)$ is read through the diagonal values, and the class of the printed realization is identified for every choice of fusion tensors. The L -symbol relations are formalized in the gauge-invariant form (4).

References

- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.

¹KleinSymmetry.omega_printedFamily_cohomologousTo, KleinSymmetry.diagonal_of_cohomologousTo_omega_printedFamily, and KleinSymmetry.kleinH3Equiv_anomalyClass_omega_printedFamily in TNLean/MPS/Examples/KleinSymmetryAnomaly.

²KleinSymmetry.lSymbol_signs_printedFamily.