

Klein Three-Cocycles: Circle and Nonzero-Complex Coefficients

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Verdict: scope-restriction (open).

1 The source statement

Ref. [GRLM23], subsection “MPO algebras representing $\mathbb{Z}_2 \times \mathbb{Z}_2$ ”, states

$$H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2^3, \quad (1)$$

with eight explicit representatives, and that the three diagonal values $\omega(g, g, g)$, $g \neq e$, of a normalized cocycle distinguish the classes. Here $U(1)$ carries the trivial action.

2 What is formalized

The formal classification uses the multiplicative group $\mathbb{C}^\times$ with the trivial action. Every three-cocycle $\omega : (\mathbb{Z}_2 \times \mathbb{Z}_2)^3 \rightarrow \mathbb{C}^\times$ is cohomologous, through a $\mathbb{C}^\times$ -valued two-cochain, to exactly one of the eight representatives, and

$$H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{C}^\times) \cong \mathbb{Z}_2^3. \quad (2)$$

Two cocycles are cohomologous exactly when their three order-two cyclic invariants $\omega(g, e, g) \omega(g, g, g)$ agree.¹

3 Missing piece and elimination plan

The comparison of (1) with (2) is not formalized in degree three; the existing circle comparison covers degree two. Two routes close it. First, the proof restricts to circle values: for a $U(1)$ -valued cocycle the normalizing two-cochain

¹`TNLean.Algebra.ScalarThreeCochain.existsUnique_cohomologousTo_kleinCocycleFamily`, `TNLean.Algebra.ScalarThreeCochain.cohomologousTo_iff_klein_three_cyclicInvariants`, and `TNLean.Algebra.ScalarThreeCochain.kleinH3Equiv` in `TNLean/Algebra/KleinCocycleCompleteness`.

$\beta(g, h) = \omega(g, e, e)/\omega(e, e, h)$, the square root used in the six-entry gauge, and the resulting two-cochain all take values in $U(1)$, and the eight representatives are ± 1 -valued. The cyclic invariants are unchanged by any $\mathbb{C}^\times$ -valued coboundary, so they also separate the $U(1)$ classes. Second, the general comparison $H^3(G, U(1)) \cong H^3(G, \mathbb{C}^\times)$ for finite G follows from $\mathbb{C}^\times \cong U(1) \times \mathbb{R}_{>0}$, since the uniquely divisible group $\mathbb{R}_{>0}$ has vanishing positive-degree cohomology for finite groups.

Verdict: scope restriction. The classification is formalized with $\mathbb{C}^\times$ coefficients; the circle-valued statement is not.

References

- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.