

The Fibonacci Module: Direct-Sum Classification

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Verdict: false-source (resolved).

1 The source claim

For a matrix product operator representing the Fibonacci fusion category $\{1, \tau; \tau \times \tau = 1 + \tau\}$, Garre-Rubio, Lootens and Molnár state that “the only solution for an invariant MPS subspace is characterized by two blocks $\{x_1, x_\tau\}$ ” with $\tau \cdot x_1 = x_\tau$ and $\tau \cdot x_\tau = x_1 + x_\tau$ [GRLM23].¹ The action multiplicities form a nonnegative integer representation of the fusion ring. When the unit acts as the identity, its Fibonacci matrix $T = M_\tau$ satisfies $T^2 = I + T$.

The printed two-block uniqueness statement does not assume indecomposability. Taken literally without that hypothesis, it is false: $F \oplus F$ is a four-block solution whenever F is the displayed regular action, since $(F \oplus F)^2 = I + (F \oplus F)$. The local correction proved here is that every nonempty indecomposable family has two blocks. Indecomposability is an added hypothesis correcting the source claim, not a condition stated in the cited passage. The unrestricted direct-sum theorem below records all decomposable solutions.

2 The earlier restriction and its removal

The earlier classification treated only one or two blocks. The finite-rank restriction is now removed: every natural matrix T satisfying $T^2 = I + T$ is a direct sum of copies of

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \quad (1)$$

No symmetry or irreducibility of T is assumed. More explicitly, let K be its finite index set and $R = \{x \in K : T_{xx} = 0\}$. There is a bijection $\sigma : K \rightarrow R \times \{0, 1\}$ such that

$$T_{xy} = \begin{cases} F_{ij}, & \sigma(x) = (r, i), \sigma(y) = (r, j), \\ 0, & \sigma(x) = (r, i), \sigma(y) = (s, j), r \neq s. \end{cases} \quad (2)$$

¹Source traceability: `Papers/2203.12563/REsubmission.tex`, lines 1991–1993.

Consequently, every unital Fibonacci nonnegative integer representation is a sum of regular representations, and every nonempty indecomposable such representation has exactly two blocks.²

The same unrestricted decomposition applies to the action coefficients of a finite family of normal tensors with positive bond dimensions, invariant under a Fibonacci matrix product operator algebra and with linearly independent periodic vectors at one positive length. It suffices that the unit fixes those vectors at that length: linear independence derives $M_1 = I$, and the symmetry theorem derives natural-number multiplicities. The zero diagonal coefficients of M_τ index the summands, and the number of blocks is twice their number.³

3 An elementary proof

The diagonal equation implies $T_{xx}^2 \leq 1 + T_{xx}$, hence $T_{xx} \in \{0, 1\}$ and $\sum_{z \neq x} T_{xz}T_{zx} = 1$. There is therefore a unique partner $y \neq x$ with $T_{xy} = T_{yx} = 1$, and $T_{xz}T_{zx} = 0$ for $z \notin \{x, y\}$. For such z , the (z, x) equation gives $T_{zy} \leq T_{zx}$, so $T_{xz}T_{zy} = 0$. The (x, y) equation reduces to $T_{xx} + T_{yy} = 1$.

Orient the pair so that $T_{xx} = 0$ and $T_{yy} = 1$. For $z \notin \{x, y\}$, the (x, z) and (y, z) equations give

$$T_{yz} \leq T_{xz}, \quad T_{xz} + T_{yz} \leq T_{yz}. \quad (3)$$

Thus both entries vanish. The partner operation is an involution: the partner of y must be x , because its outgoing row has no other nonzero off-diagonal entry. The isolated pairs partition K , proving (2). Symmetry follows from this decomposition. Each pair is a nonempty invariant subset; indecomposability forces it to be all of K .

4 Remaining distinctions

The resolved restriction concerns the rank of the nonnegative integer representation. It does not assert that an arbitrary decomposable invariant family has two blocks, nor does the matrix classification by itself identify its associativity coefficients with the Fibonacci F -symbols. The periodic-boundary reading of the symmetry hypothesis is recorded separately [con26].

References

- [con26] The TNLean contributors. Matrix product operator symmetric states: The periodic-boundary layer. Paper-gap note

²Results: `FibonacciCompression.exists_equiv_prod_of_fibNim_sq`, `FibonacciCompression.exists_equiv_prod_of_isNIMRep_fibNim`, `FibonacciCompression.fibNim_symmetric_of_sq`, and `FibonacciCompression.exists_equiv_of_isIndecomposable_fibNim`.

³Result: `FibonacciCompression.exists_equiv_prod_of_isMPOSymmetricFamily_fibNim`.

glm23_mpo_symmetric_mps_scope, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/glm23_mpo_symmetric_mps_scope.pdf.

- [GRLM23] José Garre-Rubio, Laurens Lootens, and András Molnár. Classifying phases protected by matrix product operator symmetries using matrix product states. *Quantum*, 7:927, 2023. Source coordinates in this repository use arXiv:2203.12563v3.