

Endpoint-Coordinate Scope of Truncated-Symmetry Unitarity

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Verdict: scope-restriction (open).

1 The assertion and its ambient setting

The defect construction of Ref. [FRBC25] starts from a finite group represented by simple injective matrix product unitaries. In that setting, a finite-order representing operator has zero source index, and the paper identifies the two source dimensions with the physical dimension [FRBC25, line 1547]:

$$d_r = d, \quad d_l = d. \quad (1)$$

The truncated-symmetry display (76) then defines U_g^L by closing a finite MPU string with the source triangles Y_2^g and Y_1^g . Immediately after the display, the source says that U_g^L is unitary and acts on local operators in the interior as the global symmetry does [FRBC25, lines 2062–2099].

Thus the source paragraph has two layers. First, the endpoint contraction is a unitary. Second, in the finite-group defect setting, this contraction is a spatially truncated physical symmetry after the endpoint source legs have the physical dimensions in (1).

2 The endpoint-coordinate theorem

The formal theorem proves the first layer for a single simple canonical-form-II MPU tensor.¹ For $N \geq 0$, the endpoint contraction is the matrix

$$U^{N+2} : \mathbb{C}^d \otimes (\mathbb{C}^d)^{\otimes N} \otimes \mathbb{C}^d \longrightarrow \mathbb{C}^{d_l} \otimes (\mathbb{C}^d)^{\otimes N} \otimes \mathbb{C}^{d_r}, \quad (2)$$

with entries

$$(U^{N+2})_{(l,\sigma,r),(a,\tau,b)} = \sum_{\alpha,\beta} (Y_2)_{l,(a,\alpha)} \mathcal{U}_{\alpha,\beta}^{\sigma,\tau} (Y_1)_{r,(\beta,b)}. \quad (3)$$

¹Formal declaration: `MPOTensor.IsMPUCanonicalFormII.truncatedSymmetry_isUnitary` Between in `TNLean/MPS/MPU/TruncatedSymmetryUnitarity`.

Here σ, τ are words of bulk length N . The source superscript is the total length $L = N + 2 \geq 2$, including the two endpoint triangles. The formal conclusion is unitarity between the two coordinate spaces:

$$(U^{N+2})^\dagger U^{N+2} = I_{\mathbb{C}^d \otimes (\mathbb{C}^d)^{\otimes N} \otimes \mathbb{C}^d}, \quad (4)$$

$$U^{N+2} (U^{N+2})^\dagger = I_{\mathbb{C}^{d_l} \otimes (\mathbb{C}^d)^{\otimes N} \otimes \mathbb{C}^{d_r}}. \quad (5)$$

The theorem assumes canonical form II and simplicity. The source-gate equivalence theorem for simple MPUs gives [CPGSV17, lines 532–543 and 563–601]

$$d_r d_l = d^2, \quad (6)$$

$$u^\dagger u = u u^\dagger = I. \quad (7)$$

Together with the movement-gate unitarity of (26) [FRBC25, lines 811–869 and 918–1012], these inputs prove (4) and (5) by induction on the bulk length N . The base case $N = 0$ has total length two: $U^2 = u$ after removing the empty-word coordinates. The rank product in (6) makes the two sides of (2) have equal total dimension:

$$d^{N+2} = d_l d^N d_r. \quad (8)$$

It does not by itself provide the individual endpoint identifications

$$\mathbb{C}^{d_l} \simeq \mathbb{C}^d, \quad \mathbb{C}^{d_r} \simeq \mathbb{C}^d. \quad (9)$$

3 Growth and what remains outside the theorem

The two-end source-factorization identity represented by (77) is already formalized separately. It adds one bulk site at each end:

$$(U^{N+4})_{(l, i \sigma j, r), (a, b \tau c, e)} = \sum_{t, s} (w_L)_{(l, i), (a, t)} (U^{N+2})_{(t, \sigma, s), (b, \tau, c)} (w_R)_{(j, r), (s, e)}. \quad (10)$$

The all-length unitarity proof uses the one-left-end version of this factorization, together with unitary w_L . This is a proof organization choice, not a missing two-end factorization.

The statements still outside the endpoint-coordinate theorem are different. It does not derive the finite-group endpoint dimensions (1), choose the endpoint equivalences in (9), or prove the interior action condition stated after (76). The complementary finite-chain relation is adjacent remaining scope after the movement display, not part of the literal sentence at lines 2062–2099, and the endpoint-coordinate theorem does not prove it:

$$U_g^L U_{g^{-1}}^\infty = U_{g^{-1}}^{\infty-L}. \quad (11)$$

This is (78) in Ref. [FRBC25, lines 2176–2181], and it depends on the finite-group and gauge choices preceding the defect construction.

4 Verdict and formal boundary

The formal result is a complete endpoint-coordinate unitarity theorem: it proves both Gram identities (4) and (5) for every bulk length. It is not merely an isometry, and it does not assume supplied unitarity of the source or movement gates.

It is nevertheless a scope restriction relative to the source paragraph. The source is discussing a finite-group spatial truncation of a physical symmetry, whereas the formal theorem stops at unitarity between the input space in (2) and the endpoint source-coordinate output space. Recovering the full source paragraph requires the finite-group source-rank identification, endpoint coordinate choices, the interior action condition, and the complementary relation (11). The restricted theorem may be cited as formalizing only the endpoint-coordinate unitarity part of the source assertion.

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.
- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.