

The Block Index in the Positive Generalized-Cocycle Corollary

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Verdict: local-correction (resolved).

1 Source context

Let G be a finite group acting on a set X of block labels. For a block-dependent one-cochain $\chi: X \times G \rightarrow \mathbb{C}^\times$ the generalized coboundary is

$$(d\chi)_{g,h}^x = \frac{\chi_g^{hx} \chi_h^x}{\chi_{gh}^x}, \quad (1)$$

and a block-dependent two-cochain Ω is a generalized cocycle when $d\Omega = 1$, that is, when $\Omega_{g,hk}^x \Omega_{h,k}^x = \Omega_{g,h}^{kx} \Omega_{gh,k}^x$.

The finite-group determinant lemma in Ref. [FRBC25, lines 5931–5947] starts with a block-dependent generalized cocycle $\Omega_{g,h}^x$ and constructs a block-dependent cochain $\chi_g^x = \det X_g^x$. The corollary that follows it instead asserts the existence of a positive χ_g with $\Omega = d\chi$, written without the block label [FRBC25, Corollary `cor:positive_trivial`, lines 5950–5952].

The omitted superscript is shorthand, not a restriction to a block-independent cochain. The application to ζ in Ref. [FRBC25, line 6018] concerns an ordinary scalar cochain. The later application to the block-dependent cochain $|L|$ [FRBC25, lines 6023–6029] suppresses its block label while using the generalized conclusion block by block.

2 A two-block counterexample

Let $G = C_2 = \{e, s\}$, with $s^2 = e$, act trivially on the two-element block set $X = \{a, b\}$. Define a positive block-dependent 1-cochain ρ by

$$\rho_e^a = \rho_e^b = 1, \quad \rho_s^a = 1, \quad \rho_s^b = 2, \quad (2)$$

and set $\Omega = d\rho$. Because the action is trivial, Eq. 1 reads $\Omega_{g,h}^x = \rho_g^x \rho_h^x / \rho_{gh}^x$. Every value of Ω is therefore a product and quotient of the positive numbers in

Eq. 2, hence positive; explicitly, for $x \in \{a, b\}$ and $g \in G$,

$$\Omega_{e,e}^x = \Omega_{e,g}^x = \Omega_{g,e}^x = \rho_e^x = 1, \quad (3)$$

$$\Omega_{s,s}^a = \frac{\rho_s^a \rho_s^a}{\rho_e^a} = 1, \quad (4)$$

$$\Omega_{s,s}^b = \frac{\rho_s^b \rho_s^b}{\rho_e^b} = 4. \quad (5)$$

Equations 3, 4 and 5 exhaust the four pairs (g, h) in each block, so Ω is positive.

The cocycle condition $d\Omega = d(d\rho) = 1$ holds for the generalized coboundary of any one-cochain, with no hypothesis on the action: substituting Eq. 1 into both sides of the cocycle equation gives

$$(d\rho)_{g,hk}^x (d\rho)_{h,k}^x = \frac{\rho_g^{(hk)x} \rho_{hk}^x}{\rho_{ghk}^x} \cdot \frac{\rho_h^{kx} \rho_k^x}{\rho_{hk}^x} = \frac{\rho_g^{(hk)x} \rho_h^{kx} \rho_k^x}{\rho_{ghk}^x}, \quad (6)$$

$$(d\rho)_{g,h}^{kx} (d\rho)_{gh,k}^x = \frac{\rho_g^{h(kx)} \rho_h^{kx}}{\rho_{gh}^{kx}} \cdot \frac{\rho_{gh}^{kx} \rho_k^x}{\rho_{ghk}^x} = \frac{\rho_g^{h(kx)} \rho_h^{kx} \rho_k^x}{\rho_{ghk}^x}, \quad (7)$$

and the action law $h(kx) = (hk)x$ makes Eq. 6 and Eq. 7 equal.¹

Thus Ω is a positive generalized 2-cocycle and a generalized coboundary, and it satisfies the hypotheses of the corollary. If a primitive χ_g were independent of the block, then $(d\chi)_{s,s}^x$ would also be independent of x , contradicting Eq. 5.

3 Corrected statement

The primitive in the corollary must retain its block label:

$$\Omega_{g,h}^x = (d\chi)_{g,h}^x = \frac{\chi_g^{hx} \chi_h^x}{\chi_{gh}^x}. \quad (8)$$

This agrees with the determinant construction immediately preceding the corollary and leaves all other hypotheses and conclusions unchanged.

4 Formal treatment

The formal theorem quantifies over a block-dependent cochain and proves Eq. 8.²

Verdict: resolved local correction. The surrounding determinant construction and the explicit two-block example show that the block superscript on χ is necessary.

¹Formal declaration: `TNLean.Algebra.ActionTensorGauge.isGeneralizedCocycle_coboundary` in `TNLean/Algebra/GeneralizedCocycle`.

²Formal declarations: `TNLean.Algebra.ActionTensorGauge.coboundary` and `TNLean.Algebra.LSymbol.exists_positive_coboundary` in `TNLean/Algebra/PositiveGeneralizedCocycle`.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.