

The Missing Tilde in the Inverse-Compatible Gate Argument

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Verdict: local-correction (resolved).

1 The omitted distinction

The proof of the inverse-compatible decomposition in Ref. [FRBC25, lines 5350–5487] distinguishes the original first-cut factors $\tilde{X}_1, \tilde{Y}_1$ from the candidate factors X_1, Y_1 . The distinction is explicit when the original factors are introduced and when the candidates are defined [FRBC25, lines 5391 and 5432–5435]. In the unnumbered equation immediately before the gate diagram, however, the source writes $Y_1 = K(I \otimes T^\top)X_2^\dagger$, with no tilde on the left [FRBC25, lines 5440–5443]. The gate diagram itself labels that factor $\tilde{Y}_1$ [FRBC25, lines 5444–5487]. The correction is to restore the tilde in that textual equation. A second omission of the same kind occurs in the displayed definition of the comparison K itself, where the weight of the chosen normalization is dropped; that omission and its repair are recorded in the next section. The placement of the gauge on its virtual leg must also be respected when translating the diagram into matrices; the formula below fixes that index order explicitly.

2 The two factors in matrix coordinates

Let $\mathcal{U}$ be a tensor with physical dimension d and virtual dimension D , and let T be a unitary virtual gauge such that $(\mathcal{U}^\dagger)^{ij} = T^\dagger \mathcal{U}^{ij} T$. Let ρ be a positive-definite virtual weight. Denote the first and second cut ranks by r and ℓ , respectively. Write $\tilde{X}_1, \tilde{Y}_1$ for the original first-cut factors and X_2, Y_2 for the second-cut factors. The rows of $\tilde{X}_1$ and columns of Y_2 are ordered as (i, β) ; the rows of X_2 and columns of $\tilde{Y}_1$ are ordered as (α, j) , where i, j are physical and α, β virtual indices. Their remaining indices have sizes r and ℓ , respectively. Set $A = I_d \otimes \bar{T}$ and $B = T^\top \otimes I_d$, where the bar means entrywise conjugation.

In these product-index orders, the candidate factors from (70), used in the

proof at [FRBC25, line 5432], are

$$X_1 = AY_2^\dagger, \quad (1)$$

$$Y_1 = X_2^\dagger B. \quad (2)$$

In particular, the candidate Y_1 in (2) contains no comparison matrix K . The original factors are the normalized source factors constructed using this same weight ρ , so $\tilde{X}_1^\dagger(I_d \otimes \rho)\tilde{X}_1 = I_r$; this is the normalization chosen when the factors are introduced [CPGSV17, lines 487–494]. The second omission of the argument is in the comparison itself: the source defines it without that weight, as $K = \tilde{X}_1^\dagger X_1 = \tilde{Y}_1 Y_1^\dagger$ [FRBC25, lines 5436–5439]. Under the chosen normalization the unweighted $\tilde{X}_1^\dagger$ is not a left inverse of $\tilde{X}_1$, so the printed expression is not in general a left-inverse comparison and does not satisfy $X_1 = \tilde{X}_1 K$. Restoring the weight repairs this, and gives the comparison used throughout:

$$\tilde{L}_1 = \tilde{X}_1^\dagger(I_d \otimes \rho), \quad (3)$$

$$\tilde{L}_1 \tilde{X}_1 = I_r, \quad (4)$$

$$K = \tilde{L}_1 X_1 : \mathbb{C}^\ell \longrightarrow \mathbb{C}^r. \quad (5)$$

The first-cut comparison gives $X_1 = \tilde{X}_1 K$ and $\tilde{Y}_1 = KY_1$. Hence the factor required in the gate diagram is

$$\tilde{Y}_1 = KY_1 = KX_2^\dagger B. \quad (6)$$

Equation (6) is the corrected textual assertion in the specified matrix order. It agrees with the distinction already made in Ref. [FRBC25, lines 5432–5435]. Identifying its left side with the candidate Y_1 would erase the distinction between the old and new factors. Nothing in this local correction replaces the weighted K of (5) by an unweighted overlap or identifies the inverse comparison with $K^\dagger$.

3 Where the corrected factor is used

Adjointing $AY_2^\dagger = \tilde{X}_1 K$ and using $A^\dagger A = I$ yields

$$Y_2 = K^\dagger \tilde{X}_1^\dagger A. \quad (7)$$

Use the original factors to define the two gates of (24), as in the proof diagram [FRBC25, lines 5444–5487]:

$$u_{(a,b),(i,j)} = \sum_{\beta=0}^{D-1} (Y_2)_{a,(i,\beta)} (\tilde{Y}_1)_{b,(\beta,j)}, \quad (8)$$

$$v_{(i,j),(b,a)} = \sum_{\alpha=0}^{D-1} (\tilde{X}_1)_{(i,\alpha),b} (X_2)_{(\alpha,j),a}. \quad (9)$$

Here $0 \leq a < \ell$, $0 \leq b < r$, and $0 \leq i, j < d$. For fixed i, j , put $C_{b,\gamma} = \overline{(\tilde{X}_1)_{(i,\gamma),b}}$ and $E_{\delta,a} = \overline{(X_2)_{(\delta,j),a}}$. Substituting (6) and (7) into (8) contracts the virtual gauge factors as $\overline{TT}$. If $T\overline{T} = \sigma I_D$, unitarity of T also gives $\overline{T}T = \sigma I_D$. Thus

$$u_{(a,b),(i,j)} = (K^\dagger C \overline{T} T E K^\top)_{a,b} = \sigma (K^\dagger C E K^\top)_{a,b}, \quad (10)$$

$$u = \sigma (K^\dagger \otimes K) v^\dagger. \quad (11)$$

Equation (10) locates the use of the corrected factor: its K supplies the right-hand comparison in the contraction. The row orders in (11) are (ℓ, r) for u and (r, ℓ) for $v^\dagger$; the Kronecker product maps between them without an additional swap. This is precisely the algebraic gate identity displayed by the source [FRBC25, lines 5484–5486].¹

4 An explicit check on the displayed CZX factors

The explicit factors displayed for the CZX tensor in Ref. [FRBC25, lines 4559–4659] make the role of the weight visible in coordinates. There $d = 4$, $D = 2$, the maintained gauge is the Pauli matrix Y , and the recorded weight is $\rho = I_2/2$. The displayed factors are the original ones, written $\tilde{X}_1, \tilde{Y}_1$ as in §2; the candidates of (2) built from them are $X_1 = (I_4 \otimes \tilde{Y})Y_2^\dagger$ and $Y_1 = X_2^\dagger(Y^\top \otimes I_4)$. Both candidates are the negatives of the displayed factors, and the three overlaps of Ref. [FRBC25, lines 5438–5441] take the values

$$\tilde{X}_1^\dagger X_1 = -2I_4, \quad (12)$$

$$\tilde{X}_1^\dagger (I_4 \otimes \rho) X_1 = -I_4, \quad (13)$$

$$\tilde{Y}_1 Y_1^\dagger = -I_4. \quad (14)$$

The unweighted left overlap (12) has Gram matrix $4I_4$, so it is not unitary, and it differs from the right overlap (14). The weighted comparison (13), which is the K of (5), is unitary and recovers both displayed factors as $\tilde{X}_1 = X_1 K^\dagger$ and $\tilde{Y}_1 = K Y_1$, the second being (6) itself. This is a calculation on the explicit displayed witness rather than on choice-selected factors, so it diagnoses the intermediate formula on those factors; it is not a counterexample to the proposition proved in Ref. [FRBC25, lines 5350–5487], and it changes no gauge phase.²

¹The adjointed comparison and gate contraction are recorded in `MPOTensor.sourceY_eq_inverseCompatibleComparisonK_adjoint` and `MPOTensor.sourceU_eq_smul_inverseCompatibleComparisonK_kronecker_sourceV_adjoint`, in `TNLean/MPS/MPU/InverseCompatibleGateIdentity`.

²The three overlaps, the failure of unitarity of the unweighted one, and the weighted recovery are recorded in `MPOTensor.CZX.displayedComparison_coordinates`, `MPOTensor.CZX.displayedRawComparison_not_isUnitaryBetween`, and `MPOTensor.CZX.displayedWeightedComparison_spec`, in `TNLean/MPS/MPDO/CZXComparisonNormalization`.

5 Conclusion

These are local corrections of the factor’s name and of the omitted weight, not a new hypothesis or a change of gauge. The first is justified by the already distinguished factors and by the explicit tilde in the proof diagram; the second by the source’s own normalization of the first factor, which the unweighted comparison does not invert. The elimination criterion for both is that the factor substituted into the old gate be (6), rather than the candidate alone, and that the resulting contraction retain both the weighted comparison and the scalar σ . Equations (10) and (11) meet that criterion. No broader error in the decomposition argument is asserted. In particular, the gate identity here does not by itself establish the subsequent unitarity conclusion for K or for the candidate gates.

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.
- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.